Supershielding: Confinement of Magnetic Fields

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In open-shield systems, we have derived current conditions that lead to remarkable magnetic-field confinement, which we call "supershielding." The freedom to vary both primary and shield currents is important in solving the conditions. Solutions leading to very effective shielding are illustrated by the simple system of two parallel flat strips. Generalizations include other geometries, internal shielding, and electric-field confinement. An industrial coil design serves as a reference for practical issues. We discuss mathematical limits, via a "seesaw" approach, to "perfect shielding."

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Shielding is often needed to protect the surrounding neighborhood from the fields produced by electromagnetic devices. In most applications, the shields do not completely enclose their respective systems, in order to allow access from the outside. One may assume that open shielding would not be perfect, so an outward flux leakage necessarily occurs.

In fact, for a variety of open systems of interest, we have found that magnetic flux can be surprisingly confined. Under theoretical conditions to be discussed, the magnetic field is shown to be significantly suppressed in an infinite spatial region bordering finite shields.

Consider, for example, a static magnetic field due to currents on two concentric cylindrical coils. The current distributions on both coils can be arranged to obtain excellent exterior shielding: Outside the infinite cylindrical surface in which the finite-length shield coil is imbedded, the magnetic field is strongly suppressed. This surface separates all space into two subspaces; the magnetic flux is confined to the inner subspace with little leakage into the outer subspace ("supershielding"). An industrial coil design based on the theory of this paper has been presented recently in [1].

Two-strip system.—To illustrate supershielding, we analyze two parallel flat strips whose total magnetic field can be strongly suppressed above the upper strip, provided both strip currents obey certain conditions. Analogous results will hold for other systems.

Assume the lower (upper) strip to have infinite length along the z direction, infinitesimal thickness, finite width $2L_1$ ($2L_2$) in the x direction, its center at $x = 0$, and location at $y = a_1$ ($y = a_2$) (see Fig. 1). The current density on each strip is assumed to flow only in the z direction with distributions along the x axis

$$J_i(x) = f_i(x)\delta(y - a_i)\theta(L_i - |x|), \qquad i = 1,2. \quad (1)$$

The Heaviside function θ limits the currents to the strip widths. The $f_i(x)$ are taken, for convenience, as symmetric functions. The index $i = 1$ (2) refers to the lower (upper) strip.

With the Green function from [2], the Biot-Savart formula yields the magnetic field components

$$\begin{Bmatrix} B_x(\vec r) \\ B_y(\vec r) \end{Bmatrix} = \frac{\mu_o}{2\pi}\int_0^\infty e^{k(a_2-y)}[F_2(k) + R(k)F_1(k)]\begin{Bmatrix} -\cos kx \\ \sin kx \end{Bmatrix} dk\,, \qquad y > a_2 \quad (2)$$

$$= \frac{\mu_o}{2\pi}\int_0^\infty \left(F_2(k)e^{k(y-a_2)}\begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} + e^{k(a_2-y)}R(k)F_1(k)\begin{Bmatrix} -\cos kx \\ \sin kx \end{Bmatrix}\right) dk\,, \qquad a_1 < y < a_2 \quad (3)$$

$$= \frac{\mu_o}{2\pi}\int_0^\infty e^{k(y-a_1)}[F_1(k) + R(k)F_2(k)]\begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} dk, \qquad y < a_1\,, \quad (4)$$

where $R(k) \equiv \exp(-|k|a)$, the separation between the strips is $a \equiv a_2 - a_1$, and

$$F_i(k) = \int_{-L_i}^{L_i} f_i(x)\cos kx\,dx \quad (5)$$

are the Fourier cosine transforms of the current densities on each strip. If the F_i are found first, their inverse transforms yield the strip current distributions, f_i:

$$\frac{1}{\pi}\int_0^\infty F_i(k)\cos kx\,dk = \theta(L_i - |x|)f_i(x)\,. \quad (6)$$

Supershielding conditions.—One equation for F_i is motivated by a result which immediately follows from (2) and implies perfect shielding above the upper strip:

$$\vec B = 0 \quad \text{for } y > a_2, \quad \text{if } F_2 = -RF_1\,. \quad (7)$$

Less directly, the relationship $F_2 = -RF_1$ is a sufficient condition for the vanishing of the normal component of the field along the upper strip, which also forces this component to be zero along the entire line $y = a_2$, and in fact kills the field completely above that line. (Note: $F_1 = -RF_2$ would kill the field below $y = a_1$.)

For a given $\epsilon(k)$ that sets the level of shielding, we define the set of supershielding conditions [including condition (6) for $|x| > L_i$] for the F_i:

$$F_2(k) + R(k)F_1(k) = \epsilon(k) \tag{8}$$

and

$$\int_0^\infty F_i(k)\cos kx\, dk = 0, \quad \text{for } |x| > L_i,\ i = 1,2. \tag{9}$$

"Triviality" for $L_2 < \infty$.—If we consider $\epsilon(k) = 0$ without a limit, there is an immediate worry: How can the field be exactly zero above the line $y = a_2$ even outside the upper strip and be nonzero below that line? [There is no problem, however, with continuity of the field, or any of its derivatives, across that line, beyond the strip, as can be seen from (2), (3), and (6).] By considering a moment expansion in the region above the line, we can argue that all moments for the strips as a localized source vanish and that only a trivial solution for the currents and fields exists. A related argument is to use the well-known analytic formulation of two-dimensional magnetostatics. We can analytically continue the zero-field result along paths going down below the line $y = a_2$ and avoiding the upper strip and again come to the trivial solution. Yet another argument can be based on the convolution theorem indicating that the solution $F_1 = F_2 = 0$ is forced upon us for $\epsilon(k) = 0$. But before too much weight is placed on this discussion, we note that, if the field above the upper strip is nonzero, but still small, we can have a nontrivial solution with a sizable field between the strips.

Nonzero constraints.—To consider nontrivial solutions, we impose a single nonzero field constraint between the strips, as a simple example.

Nontrivial solution for $L_2 = \infty$.—Theoretically perfect shielding [$\epsilon(k) = 0$] with smooth solutions can be found for infinite shields and reasonable nonzero field constraints. For $L_2 = \infty$, F_2 is found solely from (8), since its constraint in (9) disappears.

Nontrivial solution for $L_2 < \infty$.—This is the situation of interest: an upper strip with a finite width. We wish to satisfy (8) and (9) along with the nonzero field constraint for small, but nonzero ϵ. It is the principal point of this work that we can find numerical solutions to (8) and (9) that give us both the desired field strength inside the strips and first-rate shielding above the upper strip. Based on our numerical studies, we find:

(A) The supershielding conditions can be solved for sufficiently small ϵ to obtain significant improvements in shielding with nicely behaved current distributions. See the current and field comparisons in the figures below.

(B) The techniques used to find well-behaved solutions for these improvements involve specific constrained functionals defined so as to control current oscillations and magnitudes. See the discussion below.

(C) As a digression, there is a question about whether we can improve the (already good) shielding referred to in the above solutions. Consider approaching "perfect" shielding in a singular, limiting procedure. As $\epsilon \to 0$, the supershielding conditions, which tend to drive toward a trivial solution, will fight with the nonzero constraint. Instead of vanishing currents (as in the trivial solution), however, we may expect increasingly rapid and larger current oscillations (and the collapse of the field around the nonzero constraint) as a signal of this conflict. This suggests a "seesaw" trade-off for the step-by-step improvement of shielding by solving (8) and (9) for smaller and smaller ϵ. At each step, the current distributions will presumably become more complicated.

Supershielding procedure.—The well-behaved current distributions, giving effective shielding and referred to in (A) above, can be found with the following steps:

(1) We first impose (8) with $\epsilon = 0$. (However, the solutions at the end will correspond to $\epsilon \neq 0$.)

(2) We next assume an infinitely wide upper strip, which can support current along the whole line $y = a_2$. We enforce (9) only at discrete points $|x_j^{(i)}| > L_i$, $j = 1, \ldots, N_i$ to suppress the current beyond the strip. (For computational reasons, we also treat the lower strip in the same manner.) Now, small (oscillating) currents can reside farther out and the moment/analytical arguments no longer apply in their strong form. Finally, the solutions will be truncated to exactly satisfy (9).

(3) We define a functional for finding $F_i(k)$ and to eliminate solutions having undesirable oscillations, and in particular to smooth the behavior between the isolated points outside the strips. The above supershielding conditions plus at least one nonzero field value inside are imposed through Lagrange multipliers [3].

(4) We determine the current distributions from (6) once the transforms $F_i(k)$ are found. We verify that these are sufficiently smoothly localized whence the currents outside the desired width $2L_i$ can be neglected, and they are then truncated for $|x| < L_i$. Now $\epsilon \neq 0$ in (8) [see remark (8) for an estimate of $\epsilon(k)$.]

Numerical results for the strip model.—Consider the minimization of the following energylike functional constrained by the supershielding conditions and one nonzero x component $B_0 = -1.0$ mT imposed at the midpoint between the strips:

$$\begin{aligned} W = {} & E[F_1] + \mu_0 \int_0^\infty \lambda(k)[F_2(k) + R(k)F_1(k)]\, dk \\ & + \mu_0 \sum_{i=1}^{2} \sum_j \lambda_j^{(i)} \int_0^\infty F_i(k)\cos kx_j^{(i)}\, dk \\ & + \Lambda\left(\int_0^\infty B_x(k)F_1(k)\, dk - B_0\right), \end{aligned} \tag{10}$$

where

$$E[F_1] = \frac{\mu_0}{2}\int_0^\infty E(k)g(k)F_1^2(k)\, dk, \tag{11}$$

$$B_x(k) = -\frac{\mu_0}{\pi} R(k) \cosh\frac{ka}{2}, \tag{12}$$

$$E(k) \equiv \frac{1}{\pi} R(k) \frac{\sinh ka}{k}. \tag{13}$$

Here, $\lambda(k)$, $\lambda_j^{(i)}$, and Λ are the Lagrange multipliers for (8), (9), and the field constraint, respectively. The function $g(k)$ is described below.

The extremum for the functional (10) leads to

$$F_1(k) = -\frac{B_x(k)\Lambda}{E(k)g(k)} - \frac{\lambda_1(k)}{E(k)g(k)} + \frac{R(k)\lambda_2(k)}{E(k)g(k)}, \tag{14}$$

$$\lambda_i(k) = \sum_{j=1}^{N_i} \lambda_j^{(i)} \cos kx_j^{(i)}. \tag{15}$$

The Lagrange multipliers are found by substitution back into the constraints.

The parameters for a specific example are given in Fig. 1, except we first take a wider upper strip $L_2 = 0.15$ m. Seven points $x_j^{(i)}$ are chosen along the positive x direction where (9) is enforced. The substitution of (8) (with $\epsilon = 0$) and (14) into (9) and the field constraint yields a system of linear inhomogeneous equations for the Lagrange multipliers. We also choose $g(k) = \exp(\gamma ka)$, the nonunique factor that controls rapid oscillations and the integral convergence. [With (14), the integrand in (11) is proportional to $1/g(k)$.] With $\gamma = 0.137$, smoothly localized currents are found, truncated, and exhibited in Fig. 2. The extremely smooth and flat vanishing behavior at the strip edge of the secondary current permits us to truncate it further at $L_2 = 0.10$ m. The upper strip can thus be made narrower, as shown in Fig. 1, another surprise that suggests a useful tactic for finding solutions.

In Fig. 3, we plot the field magnitude versus x for fixed $y = 0.06$ m (i.e., one cm above the upper strip), where the magnetic field is found from the Biot-Savart formula using the currents from Fig. 2, or, after Fourier transforms, from (2)–(4). The plot (a) is the supershielding result and shows a reduction factor of at least 300 relative to the imposed value B_0 midway between the strips (and note that the field at $x = 0$ is much stronger just above the lower strip—see Fig. 1). For comparison, we show in (b) shielding obtained by the alternate method [3] where one truncates an infinite-shield current distribution, which is everywhere inferior to the supershielding result (a) by at least an order of magnitude. In (c) and (d), we consider the shielding due to the alternate method but for wider strips. The implication is that we must go to an upper strip that is at least 5 times wider than the supershield strip to achieve comparable shielding. For a picture of the field confinement due to the supershield, flux lines have been added to Fig. 1.

General remarks.—(1) Supershielding may be applied to concentric finite-length circular [1] and elliptical cylinders, slotted concentric cylinders and spheres, and other geometries in which the Laplace operator separates. (2) Complementary supershielding conditions hold where, instead of shielding the region outside the larger concentric cylinder, for example, the field inside the smaller cylinder is suppressed. (3) Supershielding occurs for both static and low-frequency magnetic *and* electric fields. In the electric-field case, the charge densities are governed by analogous sets of conditions. (4) A key idea in supershielding is that the primary and secondary current distributions have to work together. Additional degrees of freedom in the current distributions are available to establish desired field behavior in practical problems, such as uniformity over a specific spatial region (or

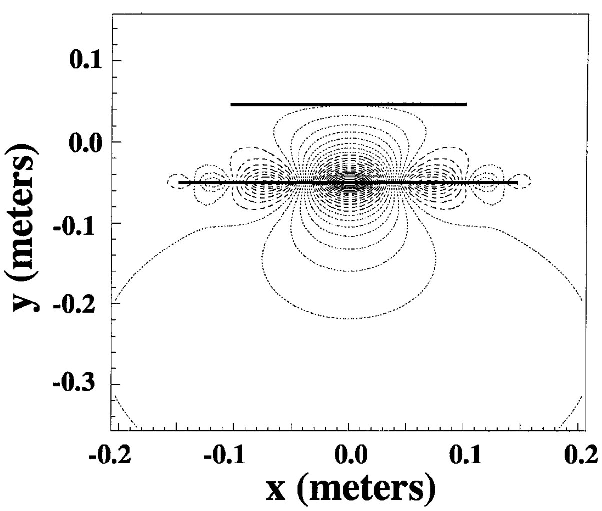

FIG. 1. Geometry of the two-strip system and the "magilla" flux lines exhibiting confinement of the total magnetic field. With a defined minimum flux density, no flux lines appear above the upper strip. For this example, $L_1 = 0.15$ m, $L_2 = 0.10$ m, and $a_2 = -a_1 = 0.05$ m.

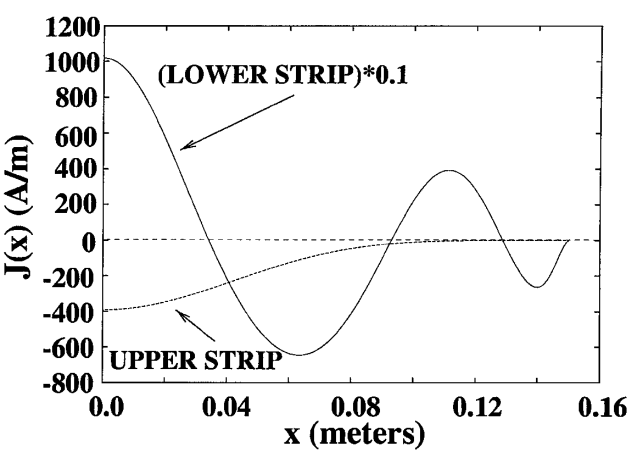

FIG. 2. Current distributions on the upper (dashed line) and lower (solid line) strips. The actual current distribution on the lower strip is a factor of 10 larger than shown. The upper strip current comes in so flatly zero at its edge that the strip (half-) width can be reduced to $L_2 = 0.10$ m.

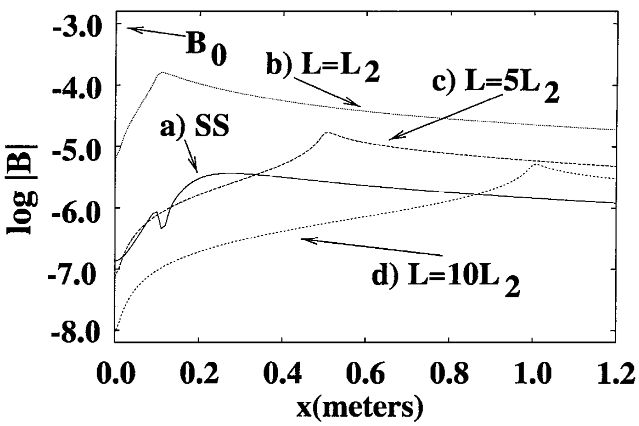

FIG. 3. Log plots for the magnitude of the magnetic field for the strip system versus x along the horizontal line $y = 0.06$ m. The supershielding result (a) is from the current distributions of Fig. 2 (but with the upper strip current truncated at $L_2 = 0.10$ m). Alternate shielding results derived from the truncation method where the current for an infinitely wide upper strip is found and then truncated to fit a given strip width equal to (b) $L_2 = 0.10$ m, (c) $5L_2$, and (d) $10L_2$. B_0 is the x component of the field at $x = y = 0$.

linearity in the practical design [1]). (5) In coil design, the continuous current distributions are approximated by thin-single-wire windings, whose density is found by stream-function techniques [1,2]. (6) The mildly oscillatory behavior in the primary strip current indicates a degree of "self-shielding." (The negative currents are achieved in practice [1,2] by reversing the single-wire winding.) However, the shielding by the secondary strip is much more important than the self-shielding by the primary strip. By a comparison with negative primary currents excluded, we compute that the primary self-shields by a factor of $1/4$ at the shield position. But the secondary strip further reduces that by a factor of about 1000. Self-shielding becomes important only when the secondary coil has a much smaller surface. (7) In view of the above remarks, let us summarize some features of our industrial design [1]. Cylindrical shields developed with previous methods [3] have at least 10 times the flux leakage of this supershielding design. Oscillations and self-shielding play only a minor role, and there is only a factor of 3 difference between the primary and secondary currents. (8) It is interesting to find a formal equation for the current distributions $f_i(x)$ explicitly. That is, a transform of (8) for $\epsilon = 0$ plus the restrictions (9) yields

$$f_2(x)\theta(L_2 - |x|) = -\frac{a}{\pi}\int_{-L_1}^{L_1} \frac{f_1(y)}{(x - y)^2 + a^2}\, dy\,, \tag{16}$$

which embodies the principal issues of the paper, is a testing ground for different mathematical techniques, and can only be solved in a limit. The extremely smooth and flat vanishing of the secondary current (which suggests bases with weighting factors like $\exp[-A/(x - L_2)^2]$ for constant A) mutes the step discontinuity. For $|x| < L_2$, we verify that (16) is satisfied with an error less than 1% of the f_2 current maximum, using the data from Fig. 2. [This is consistent with the average relative value found for $\epsilon(k)$.] A Gibbs-like undershoot beyond $L_2 = 0.15$ m is less than 2% of that maximum value, but dies off slowly, and potentially reflects some limit as to how well we can solve (16). The solutions will certainly be sensitive to the a/L_i values and the additional field constraints. (9) We observe that other factors may limit the level of shielding in a given application. With supershielding, we might achieve this practical limit before running up against a mathematical barrier. In the future, new developments in technology may make more complicated currents more feasible.

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[2] M.A. Martens, L.S. Petropoulos, R.W. Brown, J.H. Andrews, M.A. Morich, and J.L. Patrick, Rev. Sci. Instrum. **62**, 2639 (1991).

[3] Examples of using a functional approach to coil design where constraints are enforced by Lagrange multipliers have been discussed previously by R. Turner, J. Phys. E. **21**, 948 (1988); P. Mansfield and B. Chapman, J. Magn. Reson. **72**, 211 (1987); M.A. Martens, Ph.D. thesis, Case Western Reserve University, 1991. In these and later papers, an approach adequate for longer shield coils is to use constraints for infinite shields analogous to (8), with a truncation of the resulting shield current to finite lengths (the primary current is not modified in this step).