# Interacting Coherence Resonance Oscillators

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(Received 28 December 1998)

The effect of coherence resonance can change the firing process in noise-driven excitable systems towards rather regular dynamics. For such stochastic oscillators, we study the synchronization in terms of locking of the peak frequencies in the power spectrum and also in terms of phase locking. Our investigations are based on numerical simulations of coupled Morris-Lecar neuron models and on full-scale experiments with coupled monovibrator electronic circuits.

PACS numbers: 05.45.Xt, 05.40.Ca, 84.30.Ng, 87.17.Nn

During the last few decades, the interest in nonlinear science has greatly exploded as new types of oscillatory behavior, namely, chaotic and stochastic ones, have exploded. The collective behavior of systems composed of these interacting functional units can be regulated by a cooperative property, like synchronization phenomena.

For regular oscillations, when phase locking takes place, a stabilization of the phase shift between the interacting modes occurs, and the natural frequencies of oscillations become equal [1]. The classical results for regular oscillations have been generalized to some classes of chaotic oscillations. It has been shown that the synchronization in systems demonstrating the period-doubling route to chaos can be described in terms of fundamental frequency locking [2]. Following [3], synchronization of chaotic systems can be generalized to the phase synchronization.

Synchronization phenomena have also been investigated in nonlinear stochastic systems. Locking of the mean switching frequency and some kind of phase locking have been discovered both in periodically forced and in coupled noise-driven bistable systems [4,5]. Even for noisy signals, the phase description was found to be useful for the analysis of synchronization in human cardio-respiratory systems [6], for instance. These investigations are based on the classical approach to synchronization in the presence of noise [7]. The phase locking for stochastic systems is considered as an event lasting for a finite time and is described with the diffusion of phase [5] or by the shape of the phase difference distribution function [6].

Recently, a phenomenon called autonomous stochastic resonance [8] or coherence resonance (CR) [9,10] has been observed in excitable systems perturbed by noise and without external periodic forcing. Note that, in this case, a deterministic system does not exhibit any self-sustained oscillations but noise of an optimal intensity generates a quasiregular signal. Pikovsky and Kurths [9] explained the effect of CR by different noise dependences of the activation and the excursion times. Most recently, the CR effect has been confirmed by means of electronic experiments [11].

Figure 1 displays the typical shape of the power spectra in a regime of CR obtained for the relaxation-type Morris-Lecar (ML) neuron model [12] driven by the noise. Each spectrum possesses a well-defined global maximum which might be associated with the natural frequency of oscillations. The regularized behavior is observed within a reasonable range of noise intensity.

In Ref. [13], it was shown for the first time that the state of an excitable system can be described by a phase-like variable on the stochastic limit cycle. In Ref. [14], synchronous oscillation in a coupled stochastic limit cycle was observed as a collective dynamics of globally coupled identical excitable systems.

According to the above points of view, a noise-driven excitable system can be considered as a "CR oscillator" whose behavior is described by the peak frequency governed by the noise intensity and the phase introduced as the position on a stochastic limit cycle. When dealing with this new type of oscillatory unit, a question arises: To what extent will interacting nonidentical CR oscillators adjust their motion in accordance with one another so as to attain some kind of synchronization?

In this Letter, the synchronization phenomena in stochastic oscillators is investigated. The transition from nonsynchronous to synchronous state is signaled by the merging of peak frequencies in the power spectrum and also by the localized distribution of instantaneous phase differences. Our results are based on numerical

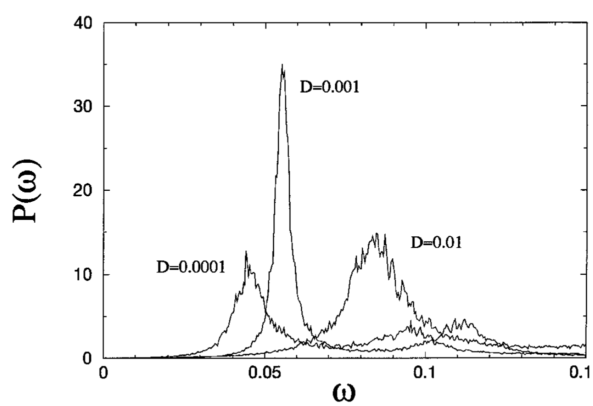


FIG. 1. The power spectra for the noise-driven Morris-Lecar model in the regime of coherence resonance.

  

simulations of the ML neuron model and on the electronic experiments with monovibrator circuits [11].

The ML model [12] is a simplified version of the Hodgkin-Huxley model which describes the spiking and refractory properties of real neurons. The diffusively coupled ML models are written as

$$\frac{dv_{1,2}}{dt} = I_{\text{ion}}(v_{1,2}, w_{1,2}) + I_{1,2} + D_{1,2}\xi_{1,2}(t) + g(v_{2,1} - v_{1,2}), \qquad (1)$$
$$\frac{dw_{1,2}}{dt} = \epsilon \frac{w_\infty(v_{1,2}) - w_{1,2}}{\tau_\infty(v_{1,2})},$$

where

$$I_{\text{ion}}(v,w) = \bar{g}_{Ca} m_\infty(v)(v_{Ca} - v) + \bar{g}_K w(v_K - v) + \bar{g}_L(v_L - v),$$
$$m_\infty(v) = 0.5\{1 + \tanh[(v + 0.01)/0.15]\},$$
$$w_\infty(v) = 0.5[1 + \tanh(v/0.3)],$$
$$\tau_\infty(v) = 1/\cosh(v/0.6).$$

Here, $v$ denotes the transmembrane voltage of a neuron, while $w$ represents the activation of the potassium current. $I$ is the external stimulus current and $\xi_{1,2}$ denote noncorrelated sources of Gaussian noise with intensity $D_{1,2}$, respectively. The last term in the first line of Eq. (1) represents the diffusive interaction with the coupling strength $g$. The parameter set used in our simulations is $I = 0.23$, $\bar{g}_{Ca} = 1.1$, $\bar{g}_K = 2.0$, $\bar{g}_L = 0.5$, $v_{Ca} = 1.0$, $v_K = -0.7$, $v_L = -0.5$ and the time separation parameter $\epsilon = 0.02$. For a detailed explanation of the parameters, see Ref. [12].

Our experimental studies are based on a monovibrator circuit, which generates a single electric impulse whenever the external signal exceeds a threshold level [11]. The electric scheme of the coupled circuits is shown in Fig. 2 and is described by

$$\tau \frac{dx_{1,2}}{dt} = \chi\{x_{1,2} - y_{1,2} - [D_{1,2}\xi_{1,2}(t) + \alpha x_{1,2} + \beta v_b]\} - y_{1,2}, \qquad (2)$$
$$\frac{dy_{1,2}}{dt} = x_{1,2} - y_{1,2} + g(x_{1,2} - y_{1,2} - x_{2,1} + y_{2,1}),$$

where $x$ and $y$ are voltages at the output of the operational amplifier and the voltage drop across $C$, respectively. The constants $\alpha$, $\beta$ are positive and defined by the value of resistors $R_1$, $R_2$, $R_3$, $R_f$. $v_b$ represents the normalized threshold voltage. The function $\chi$ is a sign function which takes values of $+1$ and $-1$ for positive and negative arguments, respectively. The noise intensities $D_{1,2}$ of the independent sources $\xi_{1,2}$ control the activation and the excursion times of each monovibrator. With vanishing mutual coupling $g = R/R_c$, CR is observed for an optimal

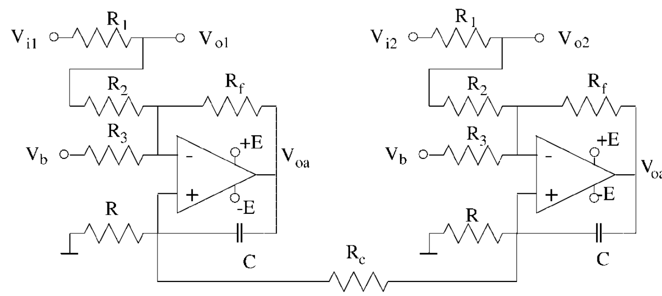


FIG. 2. The electrical scheme of the coupled monovibrator circuits. Both units are identical but the noise sources are independent.

noise [11]. Decreasing of the time scale $\tau$ provides a more prominent CR effect. Hence, CR is most pronounced in systems with strong relaxation properties.

Let us now analyze synchronization of two diffusively coupled CR oscillators. Note that the noise intensity plays the role of a control parameter governing the onset and the peak frequency of oscillations in an excitable system (Fig. 1). In this sense, the noise intensity acts like the nonlinearity and frequency parameter. To investigate the effect of frequency mismatch on the synchronization of CR oscillators, the noise intensity of the second oscillator is chosen to be different from that of the first system.

In Fig. 3a, the evolution of power spectra is plotted versus the coupling strength $g$. It is clearly seen how the peak frequencies of two oscillators approach each other and become coincident at some value of $g$. While the noise intensity $D_2$ is varied ($D_1$ and $g$ are fixed), the frequency-locked region is easily identified within a certain range of $D_2$ (Fig. 3b). The frequency-locked interval tends to become broader as the coupling strength is increased.

From the numerical simulations for the coupled ML system in Eq. (1) similar results were also observed. Instead of presenting similar plots for the simulation results, in Fig. 4, we plot the phase diagram in two dimensional parameter space of the coupling strength $g$ and the frequency mismatch parameter $D_2$. The synchronization region similar to Arnol'd tongue obtained by the condition of closeness of the peak frequencies was $\omega_1 - \omega_2 < \text{const} = 0.0002$. In the following, the instantaneous phases of two ML oscillators will be analyzed for the characterization of synchronization. This will provide us an alternative diagnosis of synchronization.

In Refs. [5,6], it has been shown how the instantaneous phases of stochastic oscillations can be locked. Once instantaneous phases are defined for the CR oscillators, similarly, it can be applied to the synchronization of two coupled CR oscillators [ML system in Eq. (1)]. According to Ref. [13], a stochastic limit cycle was defined by connecting the most likely escape trajectory out of a stationary point with the most likely return trajectory back to that point. The system's state on this circular trajectory can be described in terms of phaselike variables. Based on

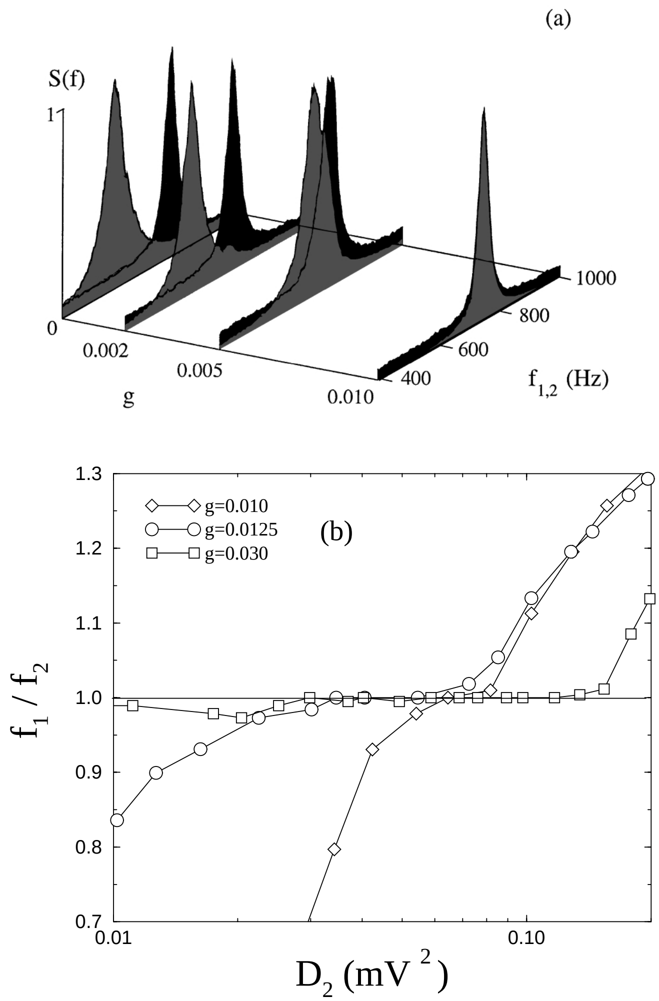


FIG. 3. The frequency locking observed in the electronic experiment: (a) the evolution of the normalized power spectrum at $D_1 = 0.1$ mV$^2$ (gray color) and $D_2 = 0.22$ mV$^2$ (black color) as the coupling strength is varied; (b) the ratio of the peak frequencies (winding number) stabilized near 1.0 value for a range of $D_2$ with $D_1 = 0.086$ mV$^2$.

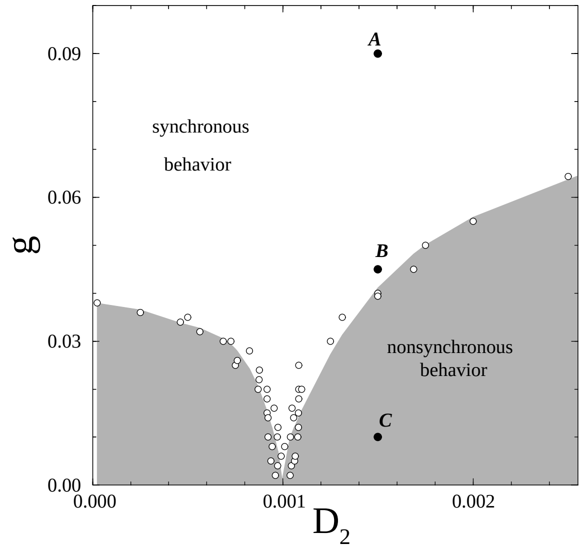


FIG. 4. The synchronization region for two coupled ML models. The noise intensity $D_2$ effectively plays the role of a frequency mismatch ($D_1 = 0.001$).

the phase variable for each ML system [15], the instantaneous phase difference is defined as $\Delta\phi = \phi_1 - \phi_2$. As the coupling is increased, for a given frequency mismatch, we observe a transition from a regime where phases rotate with different velocities ($\Delta\phi \sim \Delta\Omega t$) to a synchronous state where the phase difference looks bounded but oscillates around some mean value (Fig. 5). For strong enough coupling strength ($g = 0.08$), the phase locking for noisy systems can be observed during a long but *finite* time [5,7]. Therefore, it has to be determined *a priori* for how long the phases should be locked (on the average) to assert that a noisy system is effectively synchronized. We assume that stochastic oscillations are synchronous if no $2\pi$ phase slips occur during 50 000 periods.

Figure 6 illustrates the distribution function of phase differences (measured during the mentioned finite time) and the Poincaré sections for three discernible regimes (corresponding to the points $A$, $B$, and $C$ in Fig. 4, respectively). Inside the synchronization region (point $A$), the cross sections are concentrated in a small area (Fig. 6a) and the distribution function appears to be limited to a finite value near the zero phase difference. But outside the synchronization region (point $C$), the Poincaré section is completely different and looks like a ring in the phase space of the system (Fig. 6c) and the distribution of the phase difference is continuous over $2\pi$. At the boundary of synchronization (point $B$), the Poincaré section indicates a closed curve but it is not dense everywhere yet (Fig. 6b). These results allow us to draw an analogy between the transition from an ergodic torus to a limit cycle in deterministic case and the evolution observed in stochastic oscillations.

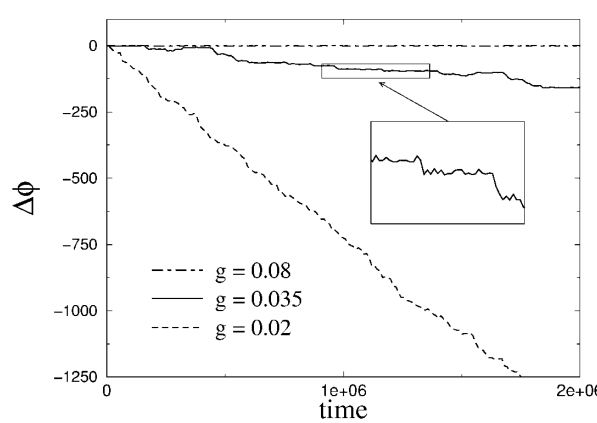


FIG. 5. The phase difference of the coupled ML model as a function of time for nonsynchronous ($g = 0.02$), nearly synchronous ($g = 0.035$), and synchronous ($g = 0.08$) states. $D_2 = 0.000\,75$. The phase slips of $2\pi$ for a nearly synchronous regime are clearly seen in the enlarged inset.

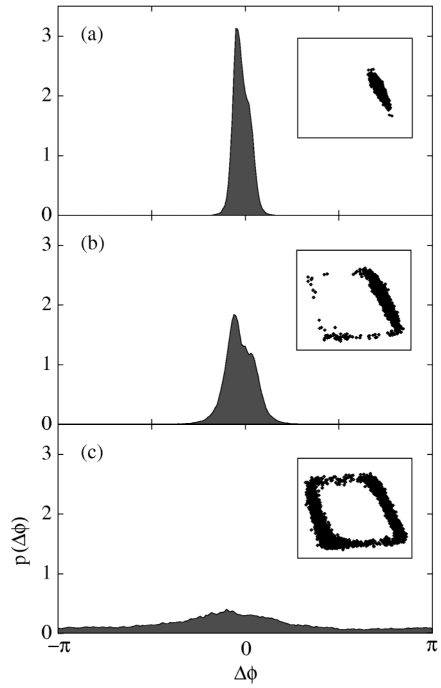


FIG. 6. The distributions of the phase difference and the Poincaré sections (insets) (a) inside the synchronization region ($g = 0.09$), (b) near the boundary ($g = 0.045$), and (c) outside this region ($g = 0.01$). $D_1 = 0.01$ and $D_2 = 0.0015$. The Poincaré section is specified by the condition $\omega_1 = 0.35$. From these plots, one can draw an analogy to the transition from a torus to a limit cycle in the deterministic case.

Following this approach we can upgrade the term "stochastic limit cycle" [13] with the notion of "stochastic torus."

In conclusion, we have considered the noise-driven excitable system in a regime of coherence resonance as a stochastic oscillator, CR oscillator which generates a rather regular signal within a certain range of noise intensity. For such coupled stochastic oscillators we have investigated the phenomenon of mutual synchronization in terms of the locking of the peak frequencies in the power spectrum and also in terms of phase locking.

Drawing an analogy to the deterministic case and following the concept of stochastic limit cycle, we have interpreted a nonsynchronous stochastic oscillation as a stochastic torus which can be characterized by the presence of two different time scales (two independent peaks in power spectra [16]) and, geometrically, as an object involving two stochastic limit cycles in the phase space of the system.

The authors thank H. Kook, S. Kim, C. Kim, A. Pikovsky, and A. Longtin for useful discussions. D. P. and O. S. are supported by KISTEP through Korea-Russia scientific exchange program during their stay at the Physics Department of Chungbuk National University. D. P. and O. S. also acknowledge support from Russian Foundation of Fundamental Research (Grant No. 99-02-17732). S. K. H. is supported by Brain Research Project of the Ministry of Science and Technology and also by the Korea Research Foundation in the program year of 1998.

*Note added.*—Recently, in Ref. [17], the role of noise in sustaining global patterns has been investigated in a locally coupled network of excitable systems.

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