Bogomol'nyi Equation for Intersecting Domain Walls

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(Received 11 June 1999)

We argue that the Wess-Zumino model with quartic superpotential admits stable static solutions in which three domain walls intersect at a junction. We derive an energy bound for such junctions and show that configurations saturating it preserve $\frac{1}{4}$ supersymmetry.

PACS numbers: 11.27.+d, 11.30.Pb

Domain walls arise in many areas of physics. They occur as solutions of scalar field theories whenever the potential is such that it has isolated degenerate minima. There are two circumstances in which this happens naturally. One is when a discrete symmetry is spontaneously broken; in this case the degeneracy is due to the symmetry. The other is when the field theory is supersymmetric; in this case the potential is derived from a superpotential, the critical points of which are degenerate minima of the potential. A simple model illustrating the latter case is the $(3 + 1)$-dimensional (or $D = 4$) Wess-Zumino (WZ) model with a superpotential $W(\phi)$ that is a polynomial function of the complex scalar field ϕ. The static domain wall solutions of this theory [1,2] are stable for topological reasons but the stability can also be deduced from the fact that a static domain wall is "supersymmetric"; i.e., it partially preserves the supersymmetry of the vacuum. An advantage of the latter point of view is that the condition for supersymmetry leads immediately to a first order "Bogomol'nyi" equation, the solutions of which automatically solve the second order field equations.

The possibilities for partial preservation of supersymmetry in the WZ model can be analyzed directly in terms of the $N = 1, D = 4$ supertranslation algebra. Allowing for all algebraically independent central charges, the matrix of anticommutators of the spinor charge components S is

$$\{S, S\} = H + \Gamma^{0i} P_i + \tfrac{1}{2}\Gamma^{0ij} U_{ij} + \tfrac{1}{2}\Gamma^{0ij}\gamma_5 V_{ij}\,, \quad (1)$$

where H is the Hamiltonian, P_i the 3-momentum, U and V are two 2-form charges, (Γ^0, Γ^i) are the 4×4 Dirac matrices, and $\gamma_5 = \Gamma^{0123}$. The fraction of supersymmetry preserved by any configuration carrying these charges is one-quarter of the number of zero-eigenvalue eigenspinors ζ of the matrix $\{S, S\}$. Supersymmetric configurations other than the vacuum will preserve either $\frac{1}{2}$ or $\frac{1}{4}$ of the supersymmetry. A domain wall in the 1-3 plane, for example, has $(U_{13}, V_{13}) = H(\cos\alpha, \sin\alpha)$ for some angle α [1,3]; the corresponding spinors ζ are eigenspinors of $\Gamma^{013}\exp(\alpha\gamma_5)$ from which it follows that the domain wall preserves $\frac{1}{2}$ supersymmetry. Now consider a configuration with nonzero H, $U_{13} = u$, $V_{23} = v$, all other charges vanishing; such a configuration preserves $\frac{1}{4}$ supersymmetry if $|u| + |v| = H$, with ζ an eigenspinor of both Γ^{013} and $\Gamma^{023}\gamma_5$. Such a configuration would naturally be associated with domain walls in the 1-3 and 2-3 planes intersecting on the 3-axis. In this paper we argue that this possibility is realized in the WZ model.

Intersections of domain walls have been extensively studied in the context of a theory with a single real scalar field φ on $\mathbb{E}^3$ [4–6]. Static configurations are presumed to satisfy an equation of the form

$$\nabla^2\varphi = V'(\varphi)\,, \quad (2)$$

where ∇^2 is the Laplacian on $\mathbb{E}^3$ and $V(\varphi)$ is a real positive function of φ with two adjacent isolated minima at which V vanishes. Let these minima be at $\varphi = \pm 1$ and let (x, y, w) be Cartesian coordinates for $\mathbb{E}^3$. If one assumes that $\varphi \to \pm 1$ as $x \to \pm\infty$, uniformly in y and w, then solutions of (2) are necessarily planar because *all* [7–11] such solutions satisfy the first order ordinary differential equation

$$\frac{d\varphi}{dx} = \sqrt{V}\,. \quad (3)$$

The solutions of this equation are the static domain walls which are stable for topological reasons. In the context of a $D = 3$ supersymmetric model they are also supersymmetric, for reasons explained at the conclusion of this Letter.

Now consider the possibility of static intersecting domain wall solutions of (2). An existence proof has been given [4] showing that (2) admits a solution representing two orthogonal domain walls. The solution has Dirichlet-type boundary data: $\varphi = 0$ on the planes $x = 0$ and $y = 0$ and $\varphi \to \pm 1$ as $|\mathbf{x}| \to \infty$ within the first quadrant. Given that the solution exists in the first quadrant, it may be obtained in the remaining quadrants by reflection. It seems clear, although we are unaware of formal proofs, that there should also exist solutions for which $2n$ domain walls intersect, adjacent walls making an angle π/n. However, all of these intersecting solutions are expected to be unstable; it is certainly the case that they cannot be supersymmetric. We shall return to this point later.

Domain walls with two or more scalar fields have been investigated in [5,6,12]. In [5,6], three-phase boundaries were shown to minimize the energy and to correspond, in the thin-wall limit, to a "Y intersection" (with 120° angles). The WZ model is a special case of models of this type. The energetics of domain wall intersections in the WZ model was investigated in [1] (see also [13]). The possibilities depend on the form of the superpotential W. If it is cubic then there are two possible domains and only one type of domain wall separating them. Intersections of two such walls cannot be more than marginally stable. Stable intersections can occur only if the superpotential is at least quartic. A quartic superpotential can therefore model a tristable medium with three possible stable domains and three types of domain wall. The $(1 + 1)$-dimensional analysis of [1] indicates that triple intersections of the three walls should be stable for some range of parameters, but static solutions representing such intersections are intrinsically $(2 + 1)$ dimensional (given that we ignore dependence on the coordinate of the string intersection), so they cannot be found from the truncation to $1 + 1$ dimensions. However, they should be minimum energy solutions of the reduction of the WZ model to $2 + 1$ dimensions. The energy density of static configurations in this reduced $(2 + 1)$-dimensional theory is

$$\mathcal{H} = \tfrac{1}{4}\nabla\phi \cdot \nabla\overline{\phi} + |W'(\phi)|^2, \tag{4}$$

where $\nabla = (\partial_x, \partial_y)$ with (x, y) being Cartesian coordinates for the two-dimensional space.

Let $z = x + iy$. The above expression for the energy density can then be rewritten as

$$\mathcal{H} = \left|\frac{\partial\phi}{\partial z} - e^{i\alpha}\overline{W}'\right|^2 + 2\,\mathrm{Re}\left(e^{-i\alpha}\frac{\partial W}{\partial z}\right) + \tfrac{1}{2}J(z,\overline{z}), \tag{5}$$

where α is an arbitrary phase, and

$$J(z,\overline{z}) = \left(\frac{\partial\phi}{\partial\overline{z}}\frac{\partial\overline{\phi}}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial\overline{\phi}}{\partial\overline{z}}\right). \tag{6}$$

We now observe that

$$Q \equiv \tfrac{1}{2}\int dx\,dy\,J(z,\overline{z}) = \int \Omega\,, \tag{7}$$

where Ω is the 2-form on 2-space induced by the closed 2-form $(i/4)\,d\overline{\phi} \wedge d\phi$ on the target space (assumed here to be the complex plane). Since Ω is real and closed, Q is a real topological charge. We may assume without loss of generality that is it non-negative. Integration over space then yields the following expression for the energy:

$$E = \int dx\,dy\left|\frac{\partial\phi}{\partial z} - e^{i\alpha}\overline{W}'\right|^2 + \mathrm{Re}[e^{-i\alpha}T] + Q\,, \tag{8}$$

where T is the complex boundary term

$$T = 2\int dx\,dy\,\frac{\partial W}{\partial z}\,. \tag{9}$$

We thereby deduce the Bogomol'nyi-type bound

$$E \geq Q + |T|\,, \tag{10}$$

which is saturated by solutions of the Bogomol'nyi equation

$$\frac{\partial\phi}{\partial z} = e^{i\alpha}\overline{W}'. \tag{11}$$

Before considering what solutions this equation may have, we shall first show that generic solutions preserve $\frac{1}{4}$ supersymmetry. The fields of the WZ model reduced to $2 + 1$ dimensions comprise a complex scalar ϕ and a complex $SL(2;\mathbb{R})$ spinor field ψ^α; we use an $SL(2;\mathbb{R})$ notation in which

$$\partial_{\alpha\beta} = \delta_{\alpha\beta}\partial_t + (\sigma_1)_{\alpha\beta}\partial_x + (\sigma_3)_{\alpha\beta}\partial_y \tag{12}$$

and $\psi_\alpha = \psi^\beta\varepsilon_{\beta\alpha}$. Similarly, $\partial^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}\partial_{\gamma\delta}$. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{8}\partial^{\alpha\beta}\phi\,\partial_{\alpha\beta}\overline{\phi} + \frac{i}{2}\overline{\psi}^\alpha\partial_{\alpha\beta}\psi^\beta + \frac{i}{2}[W''\psi^\alpha\psi_\alpha + \overline{W}''\overline{\psi}^\alpha\overline{\psi}_\alpha] - |W'|^2. \tag{13}$$

Note that the corresponding bosonic Hamiltonian density is precisely (4). The action is invariant, up to a surface term, under the infinitesimal supersymmetry transformations

$$\begin{aligned}\delta\phi &= 2i\epsilon_\alpha\psi^\alpha,\\ \delta\psi^\alpha &= -\partial^{\alpha\beta}\phi\overline{\epsilon}_\beta - 2\overline{W}'\varepsilon^{\alpha\beta}\epsilon_\beta\,,\end{aligned} \tag{14}$$

and their complex conjugates (we adopt the convention that bilinears of real spinors are pure imaginary).

We see from (14) that purely bosonic configurations are supersymmetric provided that the equation

$$\partial^{\alpha\beta}\phi\overline{\epsilon}_\beta + 2\overline{W}'\varepsilon^{\alpha\beta}\epsilon_\beta = 0 \tag{15}$$

admits a solution for some constant complex spinor ϵ. For a time-independent complex field ϕ this equation is equivalent to

$$(1 - \sigma_2)\overline{\epsilon}(\overline{\partial}\phi) + (1 + \sigma_2)\overline{\epsilon}(\partial\phi) = 2\overline{W}'\sigma_3\epsilon\,, \tag{16}$$

where $\partial \equiv \partial/\partial z$ and $\overline{\partial} \equiv \partial/\partial\overline{z}$. For a field ϕ satisfying (11) we deduce that ϵ satisfies

$$\sigma_2\overline{\epsilon} = \overline{\epsilon}, \qquad \sigma_3\overline{\epsilon} = e^{-i\alpha}\epsilon\,. \tag{17}$$

These constraints preserve just one of the four supersymmetries. Solutions of (11) are therefore $\frac{1}{4}$ supersymmetric.

The supersymmetry Noether charge of the above model is the complex $SL(2;\mathbb{R})$ spinor

$$S = \tfrac{1}{2}\int dx\,dy\{[\dot{\phi} - \sigma_1\partial_x\phi - \sigma_3\partial_y\phi]\overline{\psi} - 2i\sigma_2 W'\psi\}. \tag{18}$$

We can now use the canonical anticommutation relations of the fermion fields to compute the anticommutators. After restricting to static bosonic fields one finds that $\{S,\overline{S}\} = H - \sigma_2 Q$. Thus the junction charge Q appears as a central charge in the supertranslation algebra. The charge T appears in the $\{S,S\}$ anticommutator and the positivity of the complete matrix of supercharges implies the bound (10). Of interest here is how the junction charge Q appears in the $D = 4$ supersymmetry algebra from which we started. It appears in the same way as would the P_3 component of the momentum and is associated with the constraint $\Gamma^{03}\zeta = -\zeta$. This constraint is equivalent to $\Gamma^{023}\gamma_5\zeta = \zeta$ on the $+1$ eigenspace of Γ^{013} so, indirectly, we have found a field theory realization of the $\frac{1}{4}$ supersymmetric charge configurations that we earlier deduced from the $N = 1$, $D = 4$ supersymmetry algebra alone.

We now return to the Bogomol'nyi equation (11). If ϕ is restricted to be a function only of x then this equation reduces to the one studied in [1], which admits domain wall solutions parallel to the y axis. Each domain wall is associated with a complex topological charge of magnitude $|\int dx\, \partial_x W|$ and phase α. The question of stability of domain wall junctions was addressed in [1] by asking whether two domain walls parallel to the y axis, at least locally, can fuse to form a third domain wall of lower energy. It was found that this is possible only if their phases differ; otherwise, stability is marginal. Given that the energetics allows the formation of an intersection, we would like to find the static intersecting domain wall solution to which the system relaxes. Such solutions must depend on *both* x and y (equivalently, on both z and $\bar{z}$), and hence are much harder to find.

To simplify our task, we consider the simple quartic superpotential,

$$W(\phi) = -\tfrac{1}{4}\phi^4 + \phi\,. \tag{19}$$

This has three critical points, at $\phi = 1, \omega, \omega^2$, and a $\mathbb{Z}_3$ symmetry permuting them. There are therefore three possible domains and three types of domain wall separating them. The Bogomol'nyi equation corresponding to this superpotential is

$$\frac{\partial\phi}{\partial z} = 1 - \bar{\phi}^3. \tag{20}$$

We have set the phase $\alpha = 1$ since it can now be removed by a redefinition of z. This equation is invariant under the $\mathbb{Z}_3$ action: $(z,\phi) \to (\omega z, \omega\phi)$, so we are led to seek a $\mathbb{Z}_3$ invariant solution such that $\phi \to 1$ as one goes to infinity inside the sector $-\frac{\pi}{6} < \arg z < \frac{\pi}{6}$, subject to the condition that $\arg\phi \to \arg z$ on the boundary. By symmetry ϕ must vanish at the origin and so $\phi \approx z$ for small z. Given that a stable static triple intersection exists, there should also exist metastable networks of domain walls [14]. For example, one may imagine a static lattice consisting of hexagonal domains, rather like graphite. The vertices form triple intersections and one may consistently label the hexagons of the array with $1, \omega, \omega^2$, in such a way that no two domains which touch along a common edge carry the same label. The evolution of networks of domain walls has been studied numerically in [15]. We believe that it would be fruitful to study the WZ model in this context.

It is well known that topological defects such as strings and domain walls admit wavelike excitations traveling along them at the speed of light. The domain wall junctions considered here are no exception. One easily checks that the $D = 4$ WZ equations are satisfied if $\phi(z)$ solves our Bogomol'nyi equation (11) but is also allowed to have arbitrary dependence upon *either* $t - w$ *or* $t + w$, where w is the third space coordinate on which we reduced to get the $(2 + 1)$-dimensional model. However, *only one choice preserves supersymmetry*. To see this we note that the SL$(2;\mathbb{C})$-invariant condition for preservation of supersymmetry in the unreduced $D = 4$ WZ model is

$$\partial^{\alpha\dot\beta}\phi\bar{\epsilon}_{\dot\beta} + 2\overline{W}'\varepsilon^{\alpha\beta}\epsilon_\beta = 0\,. \tag{21}$$

Given that the reduced $D = 3$ equation (15) is satisfied, and that the spinor ϵ satisfies (17), we then deduce that

$$\partial_+\phi\bar{\epsilon} = 0\,, \tag{22}$$

where $\partial_+ = \partial_t \pm \partial_w$, the sign depending on the choice of conventions. Thus, we again have $\frac{1}{4}$ supersymmetry if $\partial_+\phi = 0$ but no supersymmetry if $\partial_-\phi = 0$. This result is not unexpected because we saw earlier that the junction charge Q appears in the $D = 4$ supersymmetry algebra in the same way as does P_3.

Note that since an individual domain wall preserves $\frac{1}{2}$ supersymmetry its low energy dynamics must be described by a $(2 + 1)$-dimensional supersymmetric field theory with two supersymmetries (corresponding to $N = 1$). The two components of the spinor field of this effective theory are the coefficients of two Nambu-Goldstone fermions associated with the broken supersymmetries. The domain wall junction preserves only $\frac{1}{4}$ supersymmetry, so there must be a total of three Nambu-Goldstone fermions localized on the intersecting domain wall configuration as a whole. Only two of these are free to propagate within the walls, so the third Nambu-Goldstone fermion must be localized on the string junction. This can also be seen by viewing the junction as a $\frac{1}{2}$-supersymmetric defect on a given wall. The fact that half of the wall's supersymmetry is preserved means that the junction's low energy dynamics is described by a $(1,0)$-supersymmetric $(1 + 1)$-dimensional field theory. This theory is chiral with one fermion that is either left moving or right moving; let us declare it to be left moving. This fermion is the Nambu-Goldstone fermion associated with the fact that the junction also breaks half the wall's supersymmetry. Its bosonic partner under $(1,0)$ supersymmetry must also be left moving. It follows that right-moving waves are supersymmetric whereas left-moving ones are not, precisely as we deduced above from other considerations.

Now that we have a good understanding of the pattern of supersymmetry breaking in the WZ model, we return to the simpler model discussed earlier with one real scalar field. This model has an $N = 1$ supersymmetrization in $2 + 1$ dimensions, with $V = 4(W')^2$, obtained by restricting all quantities in the $N = 2$ model discussed above to be real. Taking $\partial_y \varphi = 0$ we then find that solutions of (3) are supersymmetric, with the real 2-component spinor ϵ an eigenspinor of σ_3. This might seem paradoxical in view of the fact that the $N = 1$, $D = 3$ supertranslation algebra admits no central charges, of either scalar or vector type, that are *algebraically* independent of the 3-momentum. The resolution is that the anticommutator of supersymmetry charges S_α is

$$\{S_\alpha, S_\beta\} = \delta_{\alpha\beta} H + (\sigma_1)_{\alpha\beta}(P_x + T_y) + (\sigma_3)_{\alpha\beta}(P_y - T_x), \tag{23}$$

where H is the Hamiltonian, $\mathbf{P}$ is the field 2-momentum, and $\mathbf{T} = \int d^2x\, \nabla W$ is a 2-vector topological charge (the corresponding algebra of currents was discussed in [16]). For static solutions $\mathbf{P}$ vanishes, while $\partial_y W$ vanishes for solutions with $\partial_y \varphi = 0$. For such solutions we have

$$\{S, S\} = H + \sigma_3 T_x . \tag{24}$$

It follows that $H \geq |T_x|$. Field configurations that saturate this bound preserve $\frac{1}{2}$ the supersymmetry and are associated with eigenspinors of σ_3, as claimed. An intersecting domain wall solution in this $N = 1$, $(2 + 1)$-dimensional model cannot satisfy (3) (because its only static solutions are the planar domain walls) and this means that it cannot be supersymmetric. In contrast to the model with a complex scalar, one cannot use supersymmetry to argue for the stability of domain wall junctions in a model with only one real scalar field.

G. W. G. thanks Martin Barlow and Alberto Farina for informing him of their, and related, work on domain walls. We are also grateful for conversations with Paul Shellard and David Stuart.

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[14] Shortly after submission of this paper to the archives, one by Carroll, Hellerman, and Trodden appeared (hep-th/9905217) having a significant overlap with the work reported here. These authors have also pointed out to us that networks of domain walls connected by $\frac{1}{4}$ supersymmetric junctions are not themselves $\frac{1}{4}$ supersymmetric and can decay by a tunneling process. We agree. We owe the following argument to P. Saffin, who has confirmed the metastability numerically. For a given orientation, each Y junction of the model we discuss is actually one of six possible types, corresponding to the action of $\mathbb{Z}_3$ on a given Y junction and its antijunction, for which $\partial\phi/\partial z$ is replaced by $\partial\phi/\partial\bar{z}$ in (20). In a hexagonal array, each of the six junction types occurs once. Since they are each approximate solutions of different (albeit isomorphic) Bogomol’nyi equations, the network cannot be supersymmetric.

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