Anomaly Induced Effective Actions and Hawking Radiation

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The quantum stress tensor in the Unruh state for a conformal scalar propagating in a 4D Schwarzschild black hole spacetime is reconstructed in its leading behavior at infinity and near the horizon by means of an effective action derived by functionally integrating the trace anomaly.

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In the mid-1970s Hawking [1] showed that black holes are quantum mechanically unstable: they decay by the emission of thermal radiation at a temperature inversely proportional to their mass; i.e., $T_H = (8\pi M)^{-1}$ in units where $\hbar = c = G = k_B = 1$. This is one of the most astonishing discoveries of theoretical physics in the second half of the century. Nowadays black hole radiation and its thermodynamical implications, most notably the Bekenstein-Hawking area-entropy formula [2], are among the consistency tests any candidate of quantum gravity theory has to successfully pass in order to be seriously considered. Notwithstanding decades of intensive studies, the evolution and fate of an evaporating black hole (EBH) are still unknown. In the opinion of many people the final answer to this issue has to wait until a complete and self-consistent quantum gravity theory has been found. String theory appears to be the most promising candidate to achieve this goal and many efforts have been devoted to showing its compatibility with black hole radiation. However, one is still far away from understanding, within string theory, how black holes evaporate.

A more traditional field theoretical approach to the evolution of black holes driven by the quantum fluctuations of the matter fields relies on the effective action $S_{\rm eff}(g_{\alpha\beta})$. This is the so-called backreaction, which in mathematical terms is governed by the semiclassical Einstein equation

$$G_{\mu\nu}(g_{\alpha\beta}) = 8\pi\langle T_{\mu\nu}(g_{\alpha\beta})\rangle, \qquad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor for the metric $g_{\alpha\beta}$ and

$$\langle T_{\mu\nu}(g_{\alpha\beta})\rangle = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm eff}(g_{\alpha\beta})}{\delta g^{\mu\nu}} \qquad (2)$$

is the renormalized expectation value of the stress energy tensor operator for the matter fields propagating on $g_{\alpha\beta}$. A quantum state and boundary conditions appropriate to black hole evaporation have to be supplied to Eq. (1).

The framework is quantum field theory in curved spacetime [3], a semiclassical approach in which only the matter fields are quantized, whereas gravity is still described classically according to Einstein's general relativity. One expects this approximation to be consistent until the size of the EBH becomes comparable to the Planck length (10^{-33} cm). At this point one has to move to a genuine quantum gravity theory which unfortunately is still lacking. Even within semiclassical gravity, however, the evolution of an EBH is hard to follow simply because the relevant effective action $S_{\rm eff}(g_{\alpha\beta})$ is not explicitly known. The only information available on black hole evaporation comes from analytical estimates of $\langle T_{\mu\nu}\rangle$ for matter fields propagating in a fixed static Schwarzschild black hole geometry of a given mass M. Selecting a mode basis suitable for black hole evaporation (Unruh modes [4]), the matter fields are expanded in that basis, canonically quantized, and then $\langle T_{\mu\nu}\rangle$ is directly calculated by modes sum and point splitting regularization of the divergences.

Note that the Schwarzschild spacetime

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (3)$$

where $f = 1 - 2M/r$ does not satisfy the semiclassical Einstein equation (1) because the left-hand side identically vanishing unlike the right-hand side. However, one can still regard a Schwarzschild black hole as a sort of zero order (in the hole luminosity) approximation to a real EBH.

The mode basis relevant for quantization is chosen in the following way: (i) *In* modes are positive frequency with respect to Minkowski time t; (ii) *out* modes are positive frequency with respect to Kruskal $U = -4Me^{-u/4M}$, the affine parameter along the past horizon. The quantum state so defined is called the Unruh state. By condition (i) this state reduces to the usual Minkowski *in* vacuum asymptotically in the past (i.e., no incoming radiation). The condition (ii) mimics the modes coming out from a collapsing star as its surface approaches the event horizon, as shown in Hawking's original analysis [1].

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$\langle T_{\mu\nu}\rangle$ in the Unruh state has the following leading behavior at infinity [5] (only the nonzero components are shown):

$$\langle T_a{}^b\rangle \to \frac{L}{4\pi r^2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad (4)$$

$a, b = r, t$, corresponding to an outgoing flux of (approximately) blackbody radiation at the Hawking temperature T_H. L is the luminosity of the black hole and is proportional to M^{-2} (for a scalar field geometric optics yields $L = \frac{2.197\times 10^{-4}}{\pi M^2}$ [6]). On the future event horizon, $\langle T_{\mu\nu}\rangle$ is regular in a free falling frame as a consequence of (ii), and one finds [5] that $\langle T_\theta{}^\theta\rangle$ is finite and

$$\langle T_a{}^b\rangle \sim \frac{L}{4\pi(2M)^2}\begin{pmatrix} 1/f & -1 \\ 1/f^2 & -1/f \end{pmatrix}, \qquad (5)$$

describing an influx down the hole of negative energy radiation which compensates the flux escaping at infinity. From these results one expects black holes to evaporate at a rate of order M^{-2}. The evolution is then modeled as a sequence of Schwarzschild black holes with the mass parameter M decreasing along the sequence at the above rate. This should hold at least to zero order.

To go beyond this naive scheme one should directly attack the semiclassical Einstein equations: find $\langle T_{\mu\nu}(g_{\alpha\beta})\rangle$ for a sufficiently general (i.e., time dependent) EBH geometry $g_{\alpha\beta}$ and solve Eq. (1) for this geometry. This for the moment remains a dream since, as stated before, $S_{\rm eff}(g_{\alpha\beta})$ and, hence, $\langle T_{\mu\nu}(g_{\alpha\beta})\rangle$ are not known.

Significant simplifications occur when the matter fields one is considering are conformally invariant, since then at least part of $S_{\rm eff}(g_{\alpha\beta})$ can be reconstructed from the trace anomaly [3]. We shall call this part "anomaly induced effective action," i.e., $S_{\rm an}^{\rm eff}$.

At the classical level conformal invariance of the matter fields action implies a vanishing trace of the corresponding energy momentum tensor. At the quantum level, on the other hand, the renormalization procedure induces a nonvanishing expectation value of the trace which does not depend on the quantum state in which the expectation value is taken. This trace anomaly is expressed completely in terms of geometrical objects [3]

$$\langle T_\alpha^\alpha\rangle \equiv \langle T\rangle = -\frac{1}{(4\pi)^2}\left(aC^2 + bE + c\Box R\right). \qquad (6)$$

In our notation $C^2 \equiv C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ is the square of the Weyl tensor and E is an integrand of the Gauss-Bonnet topological term $E \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2$. We remark that the origin of the trace anomaly is the renormalization of the action of vacuum in a theory of conformal invariant matter fields, and that is why in (6) the R^2 term does not show up. Finally, the numerical coefficients a, b, c depend on the matter species considered [3]. The anomaly induced effective action is related to the trace anomaly by functional integration of

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta S_{\rm an}^{\rm eff}}{\delta g_{\mu\nu}} = \langle T\rangle. \qquad (7)$$

This operation allows $S_{\rm an}^{\rm eff}$ to be determined up to a Weyl invariant functional.

The basic question we would like to address in this paper is whether $S_{\rm an}^{\rm eff}$ by itself is sufficiently accurate to reproduce the basic properties of black hole evaporation and can therefore be used in the semiclassical Einstein equation (1) to get some insight in the backreaction problem. To answer this question we shall explicitly test $S_{\rm an}^{\rm eff}$ in a specific example where results can be obtained in an independent way, namely, a massless scalar field in the Unruh state propagating on a Schwarzschild black hole geometry. For this system we already know from our previous discussion the expected leading behavior of $\langle T_{\mu\nu}\rangle$ at infinity and near the horizon [see Eqs. (4) and (5)].

We shall now proceed to show that, with appropriate boundary conditions, $S_{\rm an}^{\rm eff}$ does indeed lead to a flux of radiation at infinity emitted by the Schwarzschild black hole in agreement with Eqs. (4) and (5).

We shall work with the following local form of $S_{\rm an}^{\rm eff}$ [7–9]:

$$S_{\rm an}^{\rm eff} = -\frac{c + \frac{2}{3}b}{12(4\pi)^2}\int d^4x\sqrt{-g}\,R^2 + \int d^4x\sqrt{-g}\left\{\frac{1}{2}\,\phi\Delta_4\phi + \phi\left[k_1C^2 + k_2\left(E - \frac{2}{3}\Box R\right)\right]\right\} + \int d^4x\sqrt{-g}\left(-\frac{1}{2}\,\psi\Delta_4\psi + l_1C^2\psi\right), \qquad (8)$$

where $k_1 \equiv -\frac{a}{8\pi\sqrt{-b}}$ and $k_2 \equiv \frac{\sqrt{-b}}{8\pi}$. We are considering the introduction of the auxiliary fields as a purely classical transformation which does not modify the values of a, b, c in (6). Δ_4 is the fourth order conformal operator [8]

$$\Delta_4 = \Box^2 - 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{2}{3}R\Box - \frac{1}{3}(\nabla^\mu R)\nabla_\mu, \qquad (9)$$

and l_1 is an arbitrary parameter not determined by the theory. After elimination of the auxiliary fields ϕ and ψ this expression reduces to the well-known nonlocal form given by Reigert [8] only if $l_1 = \frac{a}{8\pi\sqrt{-b}}$. For other values of l_1 this no longer happens. The difference, however, is a conformal invariant functional which, as we said, cannot be determined from the trace anomaly alone.

From Eq. (8) the equations of motion for the auxiliary fields are

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm an}^{\rm eff}}{\delta\phi} = \Delta_4\phi + k_1C^2 + k_2\left(E - \frac{2}{3}\Box R\right) = 0, \qquad (10)$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm an}^{\rm eff}}{\delta\psi} = -\Delta_4\psi + l_1 C^2 = 0\,. \qquad (11)$$

Introducing the traceless tensor $K_{\mu\nu}$ as

$$K_{\mu\nu}(\phi) = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\,\sqrt{-g}\,\{\phi\Delta_4\phi\}, \qquad (12)$$

we can write

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm an}^{\rm eff}}{\delta g^{\mu\nu}} \equiv \langle T_{\mu\nu}\rangle = K_{\mu\nu}(\phi) - K_{\mu\nu}(\psi) - 8\nabla^\lambda\nabla^\tau Z R_{\mu\lambda\nu\tau} + g_{\mu\nu} Z R^2_{\rho\sigma\alpha\beta} - 4ZR_{\mu\rho\lambda\tau}R_\nu{}^{\rho\lambda\tau}$$
$$- \frac{4k_2}{3}[(\nabla_\mu\nabla_\nu\Box\phi) - g_{\mu\nu}(\Box^2\phi)] + \ldots, \qquad (13)$$

where $Z \equiv (k_1 + k_2)\phi + l_1\psi$ and the dots indicate terms containing either the Ricci tensor $R_{\mu\nu}$ or the Ricci scalar R. Since for our subsequent analysis these terms vanish identically, they are not written in detail.

The procedure we shall adopt is to solve the equations of motion (10) and (11) for the auxiliary fields in the background Schwarzschild geometry, then insert these solutions for ϕ and ψ in $\langle T_{\mu\nu}\rangle$ of Eq. (13), and compare the results with the expected expressions Eqs. (4) and (5).

The problem we immediately have to face in trying to follow the above scheme is how to define in our framework the Unruh state, since the trace anomaly and, hence, $S_{\rm an}^{\rm eff}$ do not make any reference to a particular quantum state. Note, however, that the solution of the auxiliary field Eq. (10) [and similarly for Eq. (11)] is determined up to a solution of the homogeneous equation $\Delta_4\phi = 0$. It is through this solution that the state dependence will be encoded.

The boundary conditions that characterize the Unruh vacuum which follow from its definition [see (i) and (ii)] are as follows: (a) No incoming radiation from infinity; (b) $\langle T_{\mu\nu}\rangle$ should be regular on the future event horizon (in a free falling frame). Furthermore, in the Unruh state $\partial_t\langle T_{\mu\nu}\rangle = 0$. The homogeneous solutions of the auxiliary fields equation of motion have to implement these boundary conditions in our system if we want to correctly describe black hole evaporation.

The solution for ϕ can be given in the following general form: $\phi(r,t) = dt + w(r)$, where

$$\frac{dw}{dr} = \frac{B}{3}r + \frac{2}{3}MB - \frac{A}{6} - \frac{\alpha}{72M} + \left(\frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24}\right)\frac{1}{r-2M} - \frac{C}{2M}\frac{1}{r}$$
$$+ \ln r\left[-\frac{\alpha}{36}\frac{2M}{r(r-2M)} - \left(\frac{A}{2M} - \frac{\alpha}{48M^2}\right)\frac{r^2}{3(r-2M)}\right] + \ln(r-2M)\left[\left(\frac{A}{2M} - \frac{\alpha}{48M^2}\right)\frac{r^3-(2M)^3}{3r(r-2M)}\right], \qquad (14)$$

and we have defined $\alpha \equiv -48(k_1 + k_2)$. A, B, C, d are constants that specify the homogeneous solution. The choice of a linear t dependence appearing in Eq. (14) is the following: In the Unruh state $\langle T_{rt}\rangle \neq 0$ and this requires our field ϕ to have a time dependence otherwise $\langle T_{rt}\rangle$ would vanish identically. However, any time dependence different from the linear one would imply an explicit time dependence of $\langle T_{\mu\nu}\rangle$, which contradicts $\partial_t\langle T_{\mu\nu}\rangle = 0$. Any θ,φ dependence is forbidden by spherical symmetry.

One can express the solution for the other auxiliary field ψ in a similar form with the obvious replacements $\alpha \to \beta \equiv 48l_1$, $(A,B,C,d) \to (A',B',C',d')$. Substituting the solutions for the auxiliary fields ϕ and ψ in Eq. (13) one obtains the stress tensor $\langle T_{\mu\nu}\rangle$. We symbolically write

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}(\phi)\rangle + \langle T_{\mu\nu}(\psi)\rangle, \qquad (15)$$

dividing the contribution of each individual auxiliary field to the stress tensor. The boundary conditions (a) and (b) will be imposed on $\langle T_{\mu\nu}(\phi)\rangle$ and $\langle T_{\mu\nu}(\psi)\rangle$ separately and the physical reason for this will become clear at the end of our analysis.

Being that the calculation of $\langle T_{\mu\nu}\rangle$ is rather lengthy and boring, we shall report, here, only the basic results. A detailed analysis and discussion will be reported in a forthcoming publication [9].

The (r,t) component is the most simple to write and it reads

$$\langle T_t{}^r(\phi)\rangle = -\frac{dA}{r^2}, \qquad (16)$$

as expected from the conservation equations $\nabla^\mu\langle T_\mu{}^\nu\rangle = 0$ in the Schwarzschild spacetime under the hypothesis $\partial_t\langle T_{\mu\nu}\rangle = 0$ [10].

Examining the behavior on the horizon $r = 2M$ we have

$$\partial_r\phi \sim \frac{E}{r-2M} + \left(A - \frac{\alpha}{24M}\right)\ln(r-2M) + \text{reg}, \qquad (17)$$

where $E = -\frac{\alpha}{24} + \frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{2}{3}AM\ln 2M$.

All logarithmic divergences in $\langle T_{\mu\nu}\rangle$ are eliminated by choosing $A = \alpha/24M$ and the leading divergence on the horizon then becomes

$$\langle T_\mu{}^\nu(\phi)\rangle \sim \frac{(E^2 - 4d^2M^2)}{32M^4 f^2}\,\text{diag}(-1, 1/3, 1/3, 1/3), \qquad (18)$$

where, as usual, $f \equiv 1 - 2M/r$. This divergence vanishes if we choose $E = 2dM$ and we find

$$\langle T_a{}^b(\phi)\rangle \sim \frac{1}{(2M)^2}\begin{pmatrix} dA/f & -dA \\ dA/f^2 & -dA/f \end{pmatrix}, \qquad (19)$$

and $\langle T_\theta{}^\theta\rangle$ finite, which yields $\langle T_\mu{}^\nu\rangle$ regular on the future horizon as required by condition (b). Had we chosen $E = -2dM$, which still makes Eq. (18) vanish, the resulting $\langle T_\mu{}^\nu\rangle$ would be regular on the past horizon (and not on the future).

Examining the behavior at infinity we find that imposing $B = 0$ the leading term as $r \to \infty$ reads

$$\langle T_a{}^b(\phi)\rangle \to \frac{1}{r^2}\begin{pmatrix} -A^2/2 & -dA \\ dA & A^2/2 \end{pmatrix}, \qquad (20)$$

and $\langle T_\theta{}^\theta\rangle = 0$ at this order. Requiring no incoming radiation forces us to set $d = A/2$.

Repeating the steps for the other auxiliary field ψ we eventually arrive at the final results

$$\langle T_a{}^b\rangle \to \frac{\alpha^2 - \beta^2}{2r^2(24M)^2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad r \to \infty, \qquad (21)$$

$$\langle T_a{}^b\rangle \sim \frac{\alpha^2 - \beta^2}{2(48M^2)^2}\begin{pmatrix} 1/f & -1 \\ 1/f^2 & -1/f \end{pmatrix}, \qquad r \to 2M. \qquad (22)$$

It is remarkable that these expressions are exactly the required form of Eqs. (4) and (5) if we set $\frac{L}{4\pi} = \frac{\alpha^2-\beta^2}{2(24M)^2}$.

Before proceeding to a numerical comparison, it is rather illuminating to examine the analytic structure of the auxiliary fields once the arbitrary constants (A, B, C, d) and (A', B', C', d') are fixed according to our Unruh state conditions (a) and (b). As $r \to 2M$ we find that the condition $E = 2dM$ makes ϕ linear in v, i.e., $\phi \sim dv$ + const, which is regular on the future horizon but singular on the past horizon. On the other hand $B = 0$ and $d = A/2$ yields $\phi \sim u$ at infinity describing outgoing radiation. The same can be said for ψ. Note that this behavior emerges only as a consequence of imposing (a) and (b) separately on $\langle T_{\mu\nu}(\phi)\rangle$ and $\langle T_{\mu\nu}(\psi)\rangle$. Now, these auxiliary fields are related to the inverse of the fourth order operator Δ_4 appearing in the nonlocal form of the action (8). By our choice of constants we have therefore found the boundary conditions appropriate to the description of black hole evaporation.

We come now to the numerology. As said before, l_1 is an arbitrary parameter of our model. If it is chosen such that $S_{\rm an}^{\rm eff}$ of Eq. (8) reduces to the Reigert action [8], i.e., $l_1 = \frac{a}{8\pi\sqrt{-b}}$, inserting the appropriate values for one scalar field ($a = 1/120$, $b = -1/360$) we find $L = -\frac{1}{\pi(24M)^2}$ which is negative. This is physically meaningless. This result is analogous to the one found for minimally coupled scalar fields classically reduced to 2D under spherical symmetry [11]. On the other hand, if $l_1 = 0$ which means, by our choice of constants, $\psi = 0$ (i.e., the conformally invariant part of $S_{\rm an}^{\rm eff}$ is completely removed) one gets $L = \frac{1}{720\pi M^2}$ which differs by a factor of 6 from the result [6]. The matching of this latter would require $\beta \sim \frac{5.8\times10^{-1}}{\pi}$.

Summarizing, we have shown that the characteristic behavior at infinity and near the horizon of $\langle T_{\mu\nu}\rangle$ in the Unruh state for a Schwarzschild black hole on which our understanding of black hole evaporation so far is based can be reproduced by the anomaly induced effective action once appropriate boundary conditions are imposed on the auxiliary fields ϕ and ψ. However, one should damp enthusiasm: $S_{\rm an}^{\rm eff}$ as it stands is not able to correctly reproduce subleading terms in $\langle T_{\mu\nu}\rangle$. For example, one expects [10] leading terms in $\langle T_\theta{}^\theta\rangle$ as $r \to \infty$ to start off as r^{-4} whereas our analysis predicts the existence of a r^{-3} term. This failure is not surprising given the incompleteness of $S_{\rm an}^{\rm eff}$. In particular, it is known that the Reigert action [8] does not give the correct correlation functions of the theory [12]. It would be interesting to consider some more complicated versions of the nonlocal effective action, which are based on the Green functions of the second order conformal operators rather than on the fourth order Δ_4.

We end our paper by mentioning that a similar construction can be given also for the Hartle-Hawking state (black hole in thermal equilibrium) and for the Boulware state. We will report on this elsewhere [9].

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