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Ground-State Properties of a Rotating Bose-Einstein Condensate with Attractive Interaction

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The ground state of a rotating Bose-Einstein condensate with attractive interaction in a quasi-one-dimensional torus is studied in terms of the ratio γ of the mean-field interaction energy per particle to the single-particle energy-level spacing. The plateaus of quantized circulation are found to appear if and only if $\gamma < 1$ with the lengths of the plateaus reduced due to hybridization of the condensate over different angular-momentum states.

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The Hess-Fairbank effect [1]—disappearance of the angular momentum (AM) of liquid helium 4 as it is cooled down to absolute zero with its container kept rotating slowly—is an analog of the Meissner effect in superconductivity, and it may therefore be regarded as a hallmark of superfluidity. The requisites for the appearance of this effect are the single-valuedness of the wave function and the presence of a single Bose-Einstein condensate (BEC). Recent realization of BEC of lithium 7 [2] has opened up new possibilities associated with the attractive interaction between atoms; here the Fock exchange interaction could energetically favor the formation of hybrid BECs, which might modify the quantization of circulation and the Hess-Fairbank effect. In this Letter we investigate these possibilities in terms of the conceptually simple geometry of a quasi-one-dimensional torus.

We consider a system of N weakly interacting bosons with mass M, confined in a torus of radius R and cross-sectional area $S = \pi r^2$, where for simplicity we assume $r \ll R$. This condition justifies our assumption that the radial wave function is fixed and independent of ω—the angular frequency of rotation of the torus. At sufficiently low temperature, the interaction between dilute hard-core bosons is well approximated by Fermi's contact interaction, which is characterized by the s-wave scattering length a. The associated mean-field interaction energy per particle is given by gN, where $g = 2a\hbar^2/MRS$. The positive (negative) sign of g implies that the effective interaction between bosons is repulsive (attractive). The Hamiltonian of our system in the rotating frame is given, up to terms which are constant in our approximation, by

$$\hat{H}(\omega) = \sum_l \hbar\omega_c\left(l - \frac{\omega}{2\omega_c}\right)^2 \hat{c}_l^\dagger \hat{c}_l + \frac{g}{2}\sum_{l,m,n} \hat{c}_l^\dagger \hat{c}_m^\dagger \hat{c}_{l+n}\hat{c}_{m-n}\,, \tag{1}$$

where $\omega_c = \hbar/2MR^2$ is the critical angular frequency, l, m, and n denote the projected angular momenta in units of $\hbar$, and $\hat{c}_l^\dagger$ and $\hat{c}_l$ are the creation and annihilation operators of bosons with AM l. In Eq. (1), we have added the term $\sum_l \hbar\omega_c(\omega/2\omega_c)^2 \hat{c}_l \hat{c}_l^\dagger = N\hbar\omega^2/(4\omega_c)$ which is compensated for by the Lagrange multiplier α in Eq. (4) and therefore does not modify any result below.

We determine the minimum-energy state of the Hamiltonian (1) within a Hilbert subspace given by $|\Psi\rangle_{\rm HF} = |\ldots, n_{-l}, \ldots, n_{-1}, n_0, n_1, \ldots, n_l, \ldots\rangle$, where n_l denotes the number of bosons that occupy the state with AM l. This is nothing but the Hartree-Fock (HF) approximation; other possibilities will be discussed later. Because the total number of bosons is N, n_l's should satisfy

$$\sum_{l=-\infty}^{\infty} n_l = N\,. \tag{2}$$

The expectation value of the Hamiltonian with respect to the state $|\Psi\rangle_{\rm HF}$ is given by

$$E(\{n_l\}) = \sum_l K_l(\omega) n_l - \frac{g}{2}\sum_l n_l^2 + g\left(N^2 - \frac{N}{2}\right), \quad (3)$$

where $K_l(\omega) \equiv \hbar\omega_c(l - \omega/2\omega_c)^2$. The distribution of $\{n_l\}$ is determined so as to minimize $E(\{n_l\})$ subject to condition (2).

Case of repulsive interaction.—We first show that our ansatz wave function $|\Psi\rangle_{\rm HF}$ reproduces some well-known results. When $g > 0$, it is possible to simultaneously minimize the kinetic energy and the interaction energy in Eq. (3) with $n_l = N$ if $l = [(\omega + \omega_c)/2\omega_c]$ and $n_l = 0$ otherwise, where the symbol $[x]$ denotes the maximum integer that does not exceed x. This result implies that a single BEC is energetically favorable whether or not it is rotated; Bogoliubov's virtual-pair excitations cause only a depletion of the condensate and do not alter this conclusion. The single-valuedness of the wave function dictates that the projected AM be quantized in units of $\hbar$, but one needs something more to show that it is quantized in units of $N\hbar$. The Onsager-Feynman condition for the quantization of circulation, in fact, requires the more stringent latter condition. For the case of repulsive interaction, the Fock exchange interaction favors a single BEC [3], thereby enforcing sharp transitions between different AM states and requiring that the circulation be quantized in a uniform system as considered in this Letter. (In a related context, Castin and Dum have recently considered the stability of vortices for the nonuniform case of parabolic potentials [4]. See also Refs. [5,6].)

Case of attractive interaction.—When $g < 0$, it is impossible to simultaneously minimize the kinetic energy and the interaction energy. Were it not for the kinetic term, the lowest-energy state would be the one in which the distribution of n_l is maximally spread; no single state l could then be macroscopically occupied, and there would be no BEC. When the system is spatially confined, however, the kinetic term competes with the attractive interaction, allowing a metastable condensate to be formed.

The minimal-energy distribution $\{n_l\}$ is determined so as to minimize $E(\{n_l\})$ in Eq. (3), subject to condition (2), giving

$$n_l = \frac{N}{\gamma}[\alpha - (l - \omega/2\omega_c)^2], \quad (4)$$

where α is a Lagrange multiplier, and $\gamma \equiv |g|N/(\hbar\omega_c) = 4N|a|R/S$ is the ratio of the mean-field interaction energy per particle to the single-particle energy-level spacing. To find an estimate of γ, we rewrite it as $\gamma \sim 4 \times 10^{-4} N|a|[\text{Å}]R[\mu\text{m}]/S[\mu\text{m}^2]$. For the case of lithium 7 with $|a| \simeq 14.6$ Å, $R = 1\ \mu$m, and $r = 0.2\ \mu$m, we have $\gamma \sim 0.046N$. With suitable choice of these parameters, it is possible to prepare the system both with $\gamma < 1$ and with $\gamma > 1$.

For n_l to be positive, there must be minimum and maximum values of l, i.e., $-l_1$ and l_2. Equation (2) then becomes $\sum_{l=-l_1}^{l_2} n_l = N$, which upon substitution of Eq. (4) for n_l gives

$$\alpha = \frac{\gamma}{l_1 + l_2 + 1} + \tilde\omega(\tilde\omega + l_1 - l_2) + \frac{2(l_1^2 + l_2^2 - l_1 l_2) + l_1 + l_2}{6}, \quad (5)$$

where $\tilde\omega \equiv \omega/2\omega_c$. With the definitions of l_1 and l_2, we have $(l_1 + \tilde\omega)^2 < \alpha \le (l_1 + 1 + \tilde\omega)^2$ and $(l_2 - \tilde\omega)^2 < \alpha \le (l_2 + 1 - \tilde\omega)^2$, which lead to

$$(l_1 + l_2)\max\left\{\frac{4l_1 - 2l_2 - 1}{6} + \tilde\omega, \frac{4l_2 - 2l_1 - 1}{6} - \tilde\omega\right\} < \frac{\gamma}{l_1 + l_2 + 1} \le (l_1 + l_2 + 2)\min\left\{\frac{4l_1 - 2l_2 + 3}{6} + \tilde\omega, \frac{4l_2 - 2l_1 + 3}{6} - \tilde\omega\right\}. \quad (6)$$

These inequalities uniquely determine the pair of integers (l_1, l_2) for a given set of γ and ω.

When the torus is at rest (i.e., $\omega = 0$), Eq. (4) becomes $n_l = (N/\gamma)(\alpha - l^2)$, where $\alpha = \gamma/(2l_1 + 1) + l_1(l_1 + 1)/3$, and Eq. (6) reduces to $l_1(4l_1^2 - 1)/3 < \gamma \le (l_1 + 1)[4(l_1 + 1)^2 - 1]/3$. These inequalities uniquely determine the number $2l_1 + 1$ of macroscopically occupied AM states for a given γ. For example, $\gamma \le 1$, $1 < \gamma \le 10$, and $10 < \gamma \le 35$ correspond to $2l_1 + 1 = 1$, 3, and 5, respectively. Thus, there is a single BEC when $\gamma \le 1$. This condition agrees with the usual criterion for a metastable BEC to exist that is obtained for a parabolically confining potential using the Gross-Pitaevskii equation [7]. A new finding in our analysis is that for $\gamma > 1$ BEC becomes hybridized over different AM states.

At the continuum limit $\omega_c \to 0$ (i.e., $R \to \infty$) with $|g|N$ held constant, γ and l_1 become infinite with $\alpha/\gamma \sim O(1/l_1)$. It follows from the relation $n_l = (N/\gamma)(\alpha - l^2)$ that all $n_l's$ becomes vanishingly small, of the order of N/l_1. Thus, no BEC exists for an infinite system in accordance with the standard wisdom [3].

The analysis for the case of $\omega \ne 0$ is straightforward, and we describe here only the results that are relevant to later discussions.

(i) The region in which a single BEC with $n_l = N$ exists is given from Eq. (6) with $l_1 = -l, l_2 = l$ by

$$0 < \gamma \le -\left|\frac{\omega}{\omega_c} - 2l\right| + 1. \quad (7)$$

When ω/ω_c is an odd integer, condition (7) can never be met, so no unique BEC can exist no matter how weak the attractive interaction.

(ii) The region in which two states with AM l and $l+1$ are macroscopically occupied is given from Eq. (6) with $l_1 = -l, l_2 = l+1$ by

$$\left|\frac{\omega}{\omega_c} - 2l - 1\right| < \gamma \leq \min\left\{3\frac{\omega}{\omega_c} - 6l + 1, -3\frac{\omega}{\omega_c} + 6l + 7\right\}, \quad (8)$$

and the corresponding distribution of bosons is given by

$$n_l = \frac{N}{2}\left[1 - \frac{\omega - (2l+1)\omega_c}{\gamma\omega_c}\right], \quad n_{l+1} = N - n_l. \quad (9)$$

(iii) In general, the region in which k states with AM $l, l+1, \ldots, l+k-1$ are macroscopically occupied is given from Eq. (6) with $l_1 = -l, l_2 = l+k-1$ by

$$k(k-1)\max\left\{\frac{-6l-2k+1}{6} + \frac{\omega}{2\omega_c}, \frac{6l+4k-5}{6} - \frac{\omega}{2\omega_c}\right\} < \gamma \leq k(k+1)\min\left\{\frac{-6l-2k+5}{6} + \frac{\omega}{2\omega_c}, \frac{6l+4k-1}{6} - \frac{\omega}{2\omega_c}\right\}. \quad (10)$$

The phase diagram is shown in Fig. 1. We have thus shown that there are regions of γ and ω in which more than one AM state is macroscopically occupied. This prediction can be tested most directly by switching off the trap potential and letting the system expand. Because of Heisenberg's uncertainty relation, the tight radial confinement of the trap causes the gas to expand more rapidly in that direction than in other ones, and the superposition of BECs having different AM should result in an interference pattern with broken axisymmetry [8].

Partial quantization of circulation.—When we fix $\gamma \equiv |g|N/\hbar\omega_c < 1$ and increase ω from 0, we alternatively pass regions in which one or two AM states are macroscopically occupied (see Fig. 1), and in the latter regions the distribution of bosons between the two AM states changes continuously with ω, as can be seen from Eq. (9). What happens then to the circulation κ of the system? When a single BEC with AM l exists, κ is given by hl/M. When two AM states are macroscopically occupied, κ should be given by $h\langle l\rangle/M$, where $\langle l\rangle$ is the ensemble-averaged value of the AM. To find this value, let us restrict ourselves to the region

$$|(\omega - \omega_c)/\omega_c| < \gamma \leq -3|(\omega - \omega_c)/\omega_c| + 4, \quad (11)$$

where two states with AM $l = 0$ and $l = 1$ are macroscopically occupied, and the number of bosons in each condensate is given from Eq. (9) by $n_0 = N/2 - N(\omega - \omega_c)/(2\omega_c\gamma)$ and $n_1 = N - n_0$. Hence, the ensemble-averaged AM $\langle l\rangle$ is given by

$$\langle l\rangle = \frac{1}{2} + \frac{\omega - \omega_c}{\omega_c}\frac{\hbar\omega_c}{2|g|N}, \quad (12)$$

which does not show any sharp transition (see Fig. 2), in sharp contrast with the case of repulsive interaction. Suppose now that we perform the Hess-Fairbank experiment for the frequency of rotation and for the strength of interaction that satisfy the condition (11). Then the AM will not completely be "expelled" even at absolute zero and have a nonzero value given by Eq. (12). Only when those parameters are in the region (7) with $l = 0$, the AM should vanish at absolute zero.

Hybrid BECs vs a phase-coherent single BEC.—We have shown within the HF approximation that hybrid BECs exist for some ranges of parameters γ and ω/ω_c. Recently, Rokhsar has argued that hybrid or "fragmented" BECs are inherently unstable against the formation of a single BEC whose macroscopically occupied state is a linear combination of the "fragments" with definite relative phases [9]. In our situation, the fragments refer to macroscopically occupied AM states. We first discuss

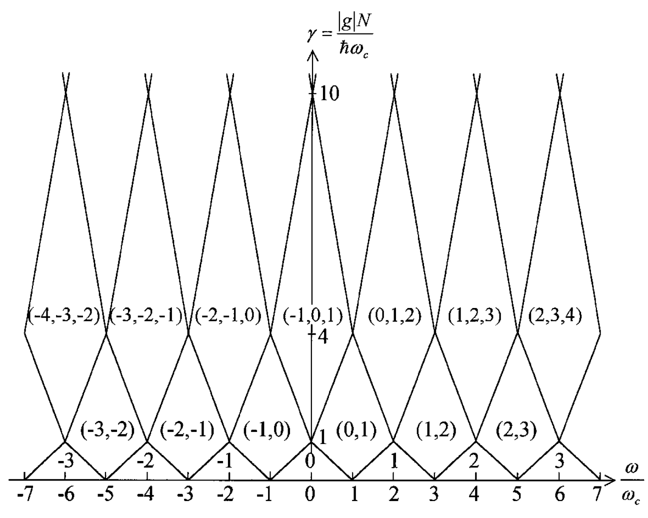

FIG. 1. Regions of ω/ω_c and $\gamma \equiv |g|N/\hbar\omega_c$ showing various phases of coexisting macroscopically occupied angular-momentum states. The triangles show regions of single Bose-Einstein condensates with AM $l = -3, -2, \ldots, 3$ from left to right. The diamond indicated by $(-3, -2)$, for example, shows the region in which two AM states $l = -3$ and -2 are macroscopically occupied.

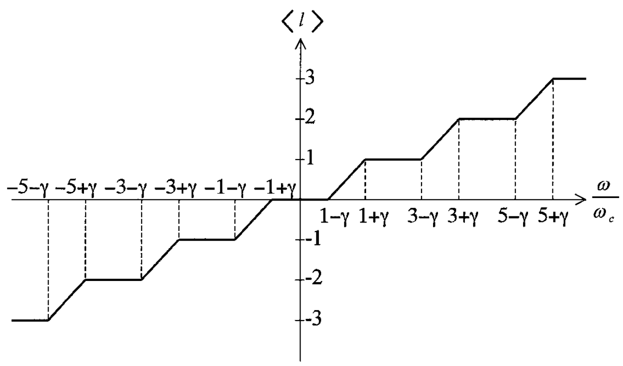

FIG. 2. Ensemble-averaged projected angular momentum $\langle l \rangle$ in units of $\hbar$ as a function of $\omega/\omega_{\rm c}$ for $\gamma \le 1$. The circulation is given by $\kappa = h\langle l \rangle/M$. The crossover regions between plateaus correspond to the regions in which two AM states are macroscopically occupied. When γ exceeds one, the plateaus disappear.

the stability of binary BECs against forming such a phase-coherent single BEC. Because the properties of the system are periodic functions of ω with periodicity $2\omega_{\rm c}$, we may consider, without loss of generality, the region (11) in which two BECs with $l = 0$ and $l = 1$ coexist. The state vector of this binary BECs is given by

$$|\Psi\rangle_{\rm HF} = |n_0, n_1\rangle = \frac{1}{\sqrt{n_0!\, n_1!}} (\hat{c}_0^\dagger)^{n_0} (\hat{c}_1^\dagger)^{n_1} |{\rm vac}\rangle, \quad (13)$$

where n_0 and n_1 are the numbers of bosons in the $l = 0$ and $l = 1$ states, which are given by Eq. (9). To be compared with this state is a single macroscopically occupied state whose creation operator $\hat{b}^\dagger$ is given by $\hat{b}^\dagger = \alpha \hat{c}_0^\dagger + \beta \hat{c}_1^\dagger$, where α and β are determined so as to minimize the total energy, subject to $|\alpha|^2 + |\beta|^2 = 1$. The corresponding single BEC is given by

$$|\Psi\rangle_{\rm single} = \frac{(\hat{b}^\dagger)^N}{\sqrt{N!}} |{\rm vac}\rangle = \frac{1}{\sqrt{N!}} (\alpha \hat{c}_0^\dagger + \beta \hat{c}_1^\dagger)^N |{\rm vac}\rangle. \quad (14)$$

The crucial observation here is that, when only two states are macroscopically occupied, the expectation value $\langle \hat{H} \rangle_{\rm single}$ of the Hamiltonian (1) over the state (14) does not contain any non-HF terms that are of the same order of magnitude as the HF terms because of the conservation of AM. Therefore, the system cannot lower its energy by establishing a relative phase coherence between the different AM states. The minimum value of $\langle \hat{H} \rangle_{\rm single}$ is reached when

$$|\alpha|^2 = \frac{1}{2}\left[1 - \frac{\omega - \omega_{\rm c}}{\gamma \omega_{\rm c}} \frac{N}{N - 1}\right], \qquad |\beta|^2 = 1 - |\alpha|^2, \quad (15)$$

and by a straightforward calculation, we find that

$$\langle \hat{H} \rangle_{\rm HF} - \langle \hat{H} \rangle_{\rm single} \simeq \frac{|g|N}{4}\left[\left(\frac{\omega - \omega_{\rm c}}{\gamma \omega_{\rm c}}\right)^2 - 1\right] < 0. \quad (16)$$

However, because the energy difference is of the order of $1/N$, the two states (13) and (14) are virtually degenerate. In real life, however, there are inhomogeneities in the container "walls" etc., which break the exact axisymmetry. Such a perturbation, however weak, could stabilize the single coherent BEC relative to a fragmented one. To show this, consider a symmetry-breaking perturbation that mixes the $l = 0$ state and the $l = 1$ state: $\hat{V} = t \hat{c}_0^\dagger \hat{c}_1 + t^* \hat{c}_1^\dagger \hat{c}_0$. It is easy to see that, while $\hat{V}$ does not lower the energy of the system for the HF state ($\langle \hat{V} \rangle_{\rm HF} = 0$), it does for the single coherent BEC; $\langle \hat{V} \rangle_{\rm single} = 2N\,{\rm Re}(\alpha^* \beta t) = -2N|\alpha^* \beta t|$, provided that $\arg\alpha - \arg\beta - \arg t = \pm\pi$. Because both $l = 0$ and $l = 1$ states are macroscopically occupied, i.e., $\alpha \sim O(1)$ and $\beta \sim O(1)$, $\langle \hat{V} \rangle_{\rm single}$ is extensive. The single coherent BEC can therefore become energetically favorable due to a (possibly infinitesimal) symmetry-breaking perturbation. It should be noted, however, that the plot of $\langle l \rangle$ versus ω in Fig. 2 remains basically unaltered because it does not depend on whether or not a phase coherence is established between two macroscopically occupied AM state.

The situation is different when more than two AM states are macroscopically occupied. Now the expectation value of the Hamiltonian contains non-HF terms that are of the same order of magnitude as the HF terms, so that without the need of the symmetry-breaking perturbation the system can lower its energy by establishing a relative phase coherence. To show this, let us consider the case of $1 + 3|\omega/\omega_{\rm c}| < \gamma < 10 - 6|\omega/\omega_{\rm c}|$, where three AM states $l = -1, 0, 1$ are macroscopically occupied. The hybrid BEC state is described by $|\Psi\rangle_{\rm HF} = |n_{-1}, n_0, n_1\rangle$ and the corresponding single coherent BEC is described by $|\Psi\rangle_{\rm single} = (\alpha \hat{c}_{-1}^\dagger + \beta \hat{c}_0^\dagger + \gamma \hat{c}_1^\dagger)^N/\sqrt{N!}\,|{\rm vac}\rangle$ with $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The expectation value of the Hamiltonian with respect to $|\Psi\rangle_{\rm HF}$ is given by

$$\begin{aligned}\langle \hat{H} \rangle_{\rm HF} = {} & K_{-1} n_{-1} + K_0 n_0 + K_1 n_1 \\ & - |g|(n_{-1} n_0 + n_0 n_1 + n_1 n_{-1}) \\ & - |g| N(N - 1)/2, \end{aligned} \quad (17)$$

which is minimized when $n_{\mp 1} = N[1 \mp (3\omega \pm \omega_{\rm c})/(\gamma \omega_{\rm c})]/3$ and $n_0 = N[1 + 2/\gamma]/3$. The expectation value of the Hamiltonian with respect to $|\Psi\rangle_{\rm single}$ is given by

$$\begin{aligned}\langle \hat{H} \rangle_{\rm single} = {} & N(K_{-1}|\alpha|^2 + K_0|\beta|^2 + K_1|\gamma|^2) - |g|N(N - 1)(|\alpha|^2|\beta|^2 + |\beta|^2|\gamma|^2 + |\gamma|^2|\alpha|^2) \\ & - |g|N(N - 1)/2 - |g|N(N - 1)(\alpha \beta^{*2} \gamma + \alpha^* \beta^2 \gamma^*). \end{aligned} \quad (18)$$

Because the last two terms are phase dependent and of the same order of magnitude as the remaining terms, it is clear that the single coherent BEC can have a lower energy than the fragmented BEC state by, e.g., the following choice of amplitudes, $\alpha = \sqrt{n_{-1}}\, e^{i\theta_\alpha}$, $\beta = \sqrt{n_0}\, e^{i\theta_\beta}$, $\gamma = \sqrt{n_1}\, e^{i\theta_\gamma}$ with the relative phase relation $\theta_\alpha - 2\theta_\beta + \theta_\gamma = 0$. Thus the ternary BEC state is unstable against the formation of a single coherent BEC. Similar mechanisms should work when more than three AM states are macroscopically occupied.

In summary, we have studied the ground-state properties of a rotating BEC with attractive interaction confined in a quasi-one-dimensional torus. When the condition (7) is met, only one AM state is macroscopically occupied. When the condition (8) is met, two BECs with different AM can, in principle, coexist. However, any deviation from the exact axisymmetry is shown to stabilize a single coherent BEC relative to a fragmented one. The plateaus of quantized circulation appear if $\gamma < 1$, but the lengths of the plateaus are reduced. In other regions of parameters γ and ω, more than two AM states are macroscopically occupied, where non-HF terms stabilize a single coherent BEC even in the presence of the exact axisymmetry.

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