Disentangling Quantum States while Preserving All Local Properties

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We consider here a disentanglement process which transforms a state ρ of two subsystems into an unentangled (i.e., separable) state, while not affecting the reduced density matrix of either subsystem. Recently, Terno [Phys. Rev. A **59**, 3320 (1999)] showed that an arbitrary state cannot be disentangled, by a physically allowable process, into a *tensor product* of its reduced density matrices. In this Letter we show that there are sets of states which can be disentangled, but only into separable states other than the product of the reduced density matrices, and other sets of states which cannot be disentangled at all. Thus, we prove that a universal disentangling machine cannot exist.

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Entanglement plays an important role in quantum physics [1,2]. Because of its peculiar nonlocal properties, entanglement is one of the main pillars of nonclassicality. The creation of entanglement and the destruction of entanglement via general operations are still under extensive study [3]. Here, we concentrate on the process of disentanglement of states. For the sake of simplicity, we concentrate on qubits in this Letter, and on the disentanglement of two subsystems.

Let there be two two-level systems "X" and "Y." The state of each such system is called a quantum bit (qubit). A pure state which is a tensor product of two qubits can always be written as $|0(X)0(Y)\rangle$ by an appropriate choice of basis, $|0\rangle$ and $|1\rangle$ for each qubit. For convenience, we drop the index of the subsystem (whenever it is possible), and order them so that X is on the left side. By an appropriate choice of the basis $|0\rangle$ and $|1\rangle$, and using the Schmidt decomposition (see [2]), an entangled pure state of two qubits can always be written as $|\psi\rangle = \cos\phi|00\rangle + \sin\phi|11\rangle$ or using a density matrix notation $\rho = |\psi\rangle\langle\psi|$,

$$\rho = [\cos\phi|00\rangle + \sin\phi|11\rangle][\cos\phi\langle 00| + \sin\phi\langle 11|]. \tag{1}$$

The reduced density matrix of each of the qubits is $\rho x = \mathrm{Tr}_Y[\rho(XY)]$ and $\rho_Y = \mathrm{Tr}_X[\rho(XY)]$. In the basis used for the Schmidt decomposition the two reduced density matrices are

$$\rho_X = \rho_Y = \begin{pmatrix} \cos^2\phi & 0 \\ 0 & \sin^2\phi \end{pmatrix}. \tag{2}$$

Following Terno [4] and Fuchs [5], let us provide the following two definitions (note that the second is an interesting special case of the first):

Definition.—"Disentanglement into a separable state" is the process that transforms a state of two (or more) subsystems into an unentangled state (in general, a mixture of product states) such that the reduced density matrices of each of the subsystems are unaffected.

Definition.—"Disentanglement into a tensor product state" is the process that transforms a state of two (or more) subsystems into a tensor product of the two reduced density matrices.

One of the main goals of this Letter is to show that a universal disentangling machine cannot exist. A universal disentangling machine is a machine that could disentangle any state which is given to it as an input. In order to prove that such a machine cannot exist, it is enough to find *one* set of states that cannot be disentangled if the data (regarding which state is used) are not available.

To analyze the process of disentanglement consider the following experiment involving two subsystems X and Y, and a sender who sends *both systems* to the receiver who wishes to disentangle the state of these two subsystems: Let the sender (Albert) and the disentangler (Nathan) define a finite set of states $|\psi_i\rangle$; let Albert choose one of the states at random, and let it be the input of the disentangling machine designed by Nathan. Nathan does not get from Albert the data regarding *which* of the states Albert chose, and Nathan's aim is to design a machine that will succeed to disentangle any of the possible states $|\psi_i\rangle$.

In the same sense that an arbitrary state cannot be cloned (a universal cloning machine does not exist [6,7]), it was recently shown by Terno [4] that an arbitrary state cannot be disentangled into a tensor product of its reduced density matrices. Note that this novel result of [4] proves that *universal disentanglement into product states* is impossible, and it leaves open the more general question of whether a *universal disentanglement* is impossible (that is, disentanglement into separable states).

We extend the investigation of the process of disentanglement beyond Terno's novel analysis in several ways. First, we show that there are nontrivial sets of states that *can* be disentangled. Then, we find a larger class (than the one found by Terno) of states which cannot be disentangled into product states. Next, we present a set of states that cannot be disentangled into tensor product states, *but* can be disentangled into separable states. Finally, we present our most important result; a set of states that *cannot be disentangled into separable states*. The existence of such a set of states proves that a universal disentangling machine cannot exist. Using the terminology of [6] we can say that our Letter shows that *a single quantum cannot be disentangled*.

Consider a set of states containing only one state. Obviously this state can be disentangled: Since the reduced density matrices of the two subsystems are known, the state should simply be replaced by the tensor product state of these reduced density matrices.

The following set of states can easily be disentangled:

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]; \qquad |\Phi_-\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]. \tag{3}$$

To disentangle them, Nathan's machine uses an ancilla which is another pair of particles in a maximally entangled state (e.g., the singlet state) in any basis. Nathan's machine swaps one of the above particles with one of the members of the added pair and traces out the ancillary particles. As a result, the state of the remaining two particles (one from each entangled pair) is

$$(1/4)[|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|], \tag{4}$$

the completely mixed state in four dimensions. This set provides a trivial example of the ability to perform the disentanglement process. It is a trivial case of disentanglement, since the two states are orthogonal: Thus, they can first be measured and distinguished, and then, once the state is known, clearly it can be disentangled.

However, exactly the same disentanglement process can be used to successfully disentangle nontrivial sets of states. Let the basis used for the two states be a different basis (and not the same basis), so the first state is still $|\Phi_+\rangle$, and the second state is

$$|\Phi'_-\rangle = \frac{1}{\sqrt{2}}[|0'0'\rangle - |1'1'\rangle]. \tag{5}$$

The same process of disentanglement still works, while now the states are nonorthogonal and cannot always be successfully distinguished. Hence, this disentanglement process is nontrivial. Note that the same process also works successfully when more than two maximally entangled states are used as the possible inputs.

We now prove that there are infinitely many sets of states that *cannot* be disentangled into product states. Our proof here follows from Terno's method, with the addition of using the Schmidt decomposition to analyze a larger class of states. The most general form of two entangled states can always be presented (by an appropriate choice of bases) as $|\psi_0\rangle = \cos\phi_0|00\rangle + \sin\phi_0|11\rangle$ and $|\psi_1\rangle = \cos\phi_1|0'0'\rangle + \sin\phi_1|1'1'\rangle$. To prove that there are states for which disentanglement into tensor product states is impossible, let us restrict ourselves to the simpler subclass

$$|\psi_0\rangle = \cos\phi|00\rangle + \sin\phi|11\rangle, \qquad |\psi_1\rangle = \cos\phi|0'0'\rangle + \sin\phi|1'1'\rangle. \tag{6}$$

There exists some basis

$$|0''\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad |1''\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7}$$

such that the bases vectors $|0\rangle$; $|1\rangle$ and $|0'\rangle$; $|1'\rangle$ become

$$|0\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}; \qquad |1\rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, \tag{8}$$

and

$$|0'\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}; \qquad |1'\rangle = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \tag{9}$$

respectively, in that basis. The states (6) are now

$$|\psi_0\rangle = c_\phi \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix}\begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix} + s_\phi \begin{pmatrix} s_\theta \\ -c_\theta \end{pmatrix}\begin{pmatrix} s_\theta \\ -c_\theta \end{pmatrix}, \qquad |\psi_1\rangle = c_\phi \begin{pmatrix} c_\theta \\ -s_\theta \end{pmatrix}\begin{pmatrix} c_\theta \\ -s_\theta \end{pmatrix} + s_\phi \begin{pmatrix} s_\theta \\ c_\theta \end{pmatrix}\begin{pmatrix} s_\theta \\ c_\theta \end{pmatrix}, \tag{10}$$

with $c_\phi \equiv \cos\phi$, etc. The overlap of the two states is $\mathrm{OL} = \langle\psi_0 | \psi_1\rangle = \cos^2 2\theta + \sin 2\phi \sin^2 2\theta$. The reduced states are given by

$$\hat{\rho}_0 = c_\phi{}^2 \begin{pmatrix} c_\theta{}^2 & c_\theta s_\theta \\ c_\theta s_\theta & s_\theta{}^2 \end{pmatrix} + s_\phi{}^2 \begin{pmatrix} s_\theta{}^2 & -c_\theta s_\theta \\ -c_\theta s_\theta & c_\theta{}^2 \end{pmatrix}, \qquad \hat{\rho}_1 = c_\theta{}^2 \begin{pmatrix} c_\theta{}^2 & -c_\theta s_\theta \\ -c_\theta s_\theta & s_\theta{}^2 \end{pmatrix} + s_\phi{}^2 \begin{pmatrix} s_\theta{}^2 & c_\theta s_\theta \\ c_\theta s_\theta & c_\theta{}^2 \end{pmatrix}. \tag{11}$$

Thus, the state after the disentanglement into tensor product states is $(\rho_{\mathrm{disent}})_0 = \hat{\rho}_0\hat{\rho}_0$ or $(\rho_{\mathrm{disent}})_1 = \hat{\rho}_1\hat{\rho}_1$.

The minimal probability of error for distinguishing two states [8] is given by $\mathrm{PE} = \frac{1}{2} - \frac{1}{4}\mathrm{Tr}|\rho_0 - \rho_1|$. For two pure states there is a simpler expression: $\mathrm{PE} = \frac{1}{2} - \frac{1}{2}\sqrt{[1 - OL^2]}$. Thus,

$$\mathrm{PE}_{\mathrm{ent}} = \frac{1}{2} - \frac{1}{2}\sqrt{[1 - (c_{2\theta}{}^2 + s_{2\phi}s_{2\theta}{}^2)^2]} \tag{12}$$

for the two initial entangled states. This probability of error is minimal, hence it cannot be reduced by any physical process. Therefore, if, for some θ and ϕ, the disentanglement into the tensor product states *reduces* the PE, then that process is nonphysical.

The difference of the states obtained after disentangling into tensor product states is $\Delta_{\mathrm{disent}} = \hat{\rho}_0\hat{\rho}_0 - \hat{\rho}_1\hat{\rho}_1$. This matrix is

$$\Delta_{\mathrm{disent}} = \cos 2\phi \sin 2\theta \begin{pmatrix} 0 & a & a & 0 \\ a & 0 & 0 & b \\ a & 0 & 0 & b \\ 0 & b & b & 0 \end{pmatrix}, \tag{13}$$

with $a = \cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta$ and $b = \cos^2\phi \times$

$\sin^2\theta + \sin^2\phi\cos^2\theta$. After diagonalization, we can calculate the trace-norm, so finally we get

$$\begin{aligned} \mathrm{PE}_{\mathrm{disent}} &= \frac{1}{2} - \frac{1}{\sqrt{2}}\sin 2\theta\cos 2\phi\sqrt{a^2 + b^2}\,, \\ &= \frac{1}{2} - \frac{1}{2}\, s_{2\theta}c_{2\phi}\sqrt{1 + c_{2\phi}{}^2 c_{2\theta}{}^2}\,. \end{aligned} \tag{14}$$

We can now observe that there are values of θ and ϕ, e.g., $\theta = \phi = \pi/8$, for which the outcomes of the disentanglement process are illegitimate since they satisfy $\mathrm{PE}_{\mathrm{disent}} < \mathrm{PE}_{\mathrm{ent}}$. Once these outcomes are illegitimate the disentanglement process leading to these outcomes is nonphysical, proving that a disentangling machine which disentangles the states $|\psi_0\rangle$ and $|\psi_1\rangle$ cannot exist for these values of θ and ϕ. Therefore, this analysis provides a proof (similar to Terno's proof [4]) that a universal machine performing disentanglement into tensor product states cannot exist.

The tools used so far (to analyze disentanglement into tensor product states) are not easily generalized to the case of disentanglement into separable states, since in the latter case the output is not unique. Luckily, much simpler tools are found here to be sufficient. Let us recall some proofs of the no-cloning argument, since the methods we shall use here are quite similar to those used in the no-cloning argument. Let the cloner obtain an unknown state and try to clone it. To prove that this is impossible, it is enough to provide one set of states for which the cloner cannot clone an arbitrary state in this set. Let the sender and the cloner use three states $|0\rangle$, $|1\rangle$, and $|+\rangle = (1/\sqrt{2})[|0\rangle + |1\rangle]$. The most general process which can be used here in the attempt to clone the unknown state from this set is to attach an ancilla in an arbitrary dimension and in a known state (say $|E\rangle$), to transform the entire system using an arbitrary unitary transformation, and to trace out the unrequired parts of the ancilla. In order to clone the states $|0\rangle$ and $|1\rangle$ the transformations are restricted to be

$$|E0\rangle \to |E_0 00\rangle; \qquad |E1\rangle \to |E_1 11\rangle\,, \tag{15}$$

and once the remaining ancilla is traced out, the cloning process is completed. Because of linearity, this fully determines the transformation of the last state to be

$$|E+\rangle \to \frac{1}{\sqrt{2}}[|E_0 00\rangle + |E_1 11\rangle]\,, \tag{16}$$

while a cloning process should yield

$$|E+\rangle \to |E_+ ++\rangle\,. \tag{17}$$

The contradiction is clearly apparent since, once the remaining ancilla is traced out, the second expression has a nonzero amplitude for the term $|01\rangle$ while the first expression does not. The conventional way [6] of proving the no-cloning theorem (using only two states, say $|0\rangle$, and $|+\rangle$) is to compare the overlap before and after the transformation (it must be equal due to the unitarity of quantum mechanics): We obtain that $|\langle E\,|\,E\rangle\langle 0\,|\,+\rangle| = |\langle E_0\,|\,E_+\rangle\langle 0\,|\,+\rangle\langle 0\,|\,+\rangle|$. Hence $1 = |\langle E_0\,|\,E_+\rangle\langle 0\,|\,+\rangle|$ which is obviously wrong since $|\langle E_0\,|\,E_+\rangle| \le 1$ and $|\langle 0\,|\,+\rangle| < 1$.

We shall now use the linearity of quantum mechanics to show that there are states that cannot be disentangled into tensor product states, but can only be disentangled into a mixture of tensor product states. Surprisingly, our proof is mainly based on the *disentanglement of product states,* that is, on the disentanglement of states which are not even entangled before the disentanglement process. The reason for the usefulness of such states is that they provide rigid restrictions on the allowed transformations: When a successful disentanglement is applied onto any pure product state, the state must be left unmodified. That is, up to an overall phase,

$$|00\rangle \to |00\rangle \tag{18}$$

(in an appropriate basis).

The following set of states cannot be disentangled into product states:

$$\begin{aligned} |\psi_0\rangle &= |00\rangle\,, \\ |\psi_1\rangle &= |11\rangle\,, \\ |\psi_2\rangle &= |00\rangle + |11\rangle\,. \end{aligned} \tag{19}$$

We shall assume that these states can be disentangled into product states and we shall reach a contradiction. Note that the resulting states should be $|\psi_0\rangle$ and $|\psi_1\rangle$ in the first two cases [see Eq. (18)], and the resulting state should be the completely mixed state (in four dimensions) in the last case [see Eq. (4)].

The most general process which can be used here is to attach an ancilla in an arbitrary dimension and in a known state (say $|E\rangle$), to transform the entire system using an arbitrary unitary transformation, and to trace out the ancilla. In order to avoid changing the states $|\psi_0\rangle$ and $|\psi_1\rangle$, the transformations are restricted to be

$$\begin{aligned} |E\psi_0\rangle &= |E00\rangle \to |E_0 00\rangle\,; \\ |E\psi_1\rangle &= |E11\rangle \to |E_1 11\rangle\,. \end{aligned} \tag{20}$$

As in the no-cloning argument, these transformations fully determine the transformation of the last state to be

$$|E\psi_2\rangle \to \frac{1}{\sqrt{2}}[|E_0 00\rangle + |E_1 11\rangle]\,. \tag{21}$$

Once we trace out the ancilla, the resulting state is still entangled unless $|E_0\rangle$ and $|E_1\rangle$ are orthogonal. The proof of that statement is as follows: Without loss of generality the states $|E_0\rangle$ and $|E_1\rangle$ can be written as $|E_0\rangle = |0\rangle$ and $|E_1\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha^2| + |\beta^2| = 1$. Thus,

$|E\psi_2\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle(|00\rangle + \alpha|11\rangle) + \frac{\beta}{\sqrt{2}}|111\rangle$. When the ancilla is traced out the remaining state is

$$\begin{pmatrix} 1/2 & 0 & 0 & \alpha^*/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha/2 & 0 & 0 & 1/2 \end{pmatrix}. \tag{22}$$

The resulting state is entangled unless $\alpha = 0$. Thus, in a successful disentanglement process $\alpha = 0$ and hence, $|E_1\rangle = e^{i\theta}|1\rangle$. The resulting state, however, is not a tensor product state, thus the above set of states *cannot* be disentangled into tensor product states.

At the same time, this example also shows that the above set of states *can* be disentangled into a mixture of tensor product states. The state (22) still has the correct reduced density matrices for each subsystem—the completely mixed state in two dimensions. With $\alpha = 0$, the resulting state is $(1/2)[|00\rangle\langle 00| + |11\rangle\langle 11|]$, so we succeeded in showing an example where the states can only be disentangled into a separable state, but not into a tensor product state.

Our result resembles a result regarding two commuting mixed states [9]: these states cannot be cloned, but they can be broadcast. That is, the resulting state of the cloning device cannot be a tensor product of states which are equal to the original states, but can be a separable state whose reduced density matrices are equal to the original states [10].

At this point, the main question (raised by [4] and [5]) is still left open: Can there be a universal disentangling machine? That is, can there exist a machine that disentangles any set of states into separable states? We shall now show that such a machine cannot exist.

Our result is obtained by combining several of the previous techniques: the use of linearity, unitarity, and the disentanglement of product states.

Consider the following set of states:

$$\begin{aligned} |\psi_0\rangle &= |00\rangle, \\ |\psi_1\rangle &= |11\rangle, \\ |\psi_2\rangle &= (1/\sqrt{2})[|00\rangle + |11\rangle], \\ |\psi_3\rangle &= |++\rangle, \end{aligned} \tag{23}$$

in which we added the state $|\psi_3\rangle$ to the previous set. This set of states cannot be disentangled even into separable states.

The allowed transformations are now more restricted since, in addition to [Eq. (20)], the state $|\psi_3\rangle$ must also not be changed by the disentangling machine,

$$|E\psi_3\rangle = |E{+}{+}\rangle \rightarrow |E_+{+}{+}\rangle. \tag{24}$$

Because of unitarity, we obtain $|\langle E|E\rangle\langle 0|+\rangle\langle 0|+\rangle| = |\langle E_0|E_+\rangle\langle 0|+\rangle\langle 0|+\rangle|$, and also $|\langle E|E\rangle\langle 1|+\rangle \times \langle 1|+\rangle| = |\langle E_1|E_+\rangle\langle 1|+\rangle\langle 1|+\rangle|$. These expressions yield $1 = |\langle E_0|E_+\rangle|$, and $1 = |\langle E_1|E_+\rangle|$, from which we conclude that $e^{i\chi}|E_0\rangle = |E_1\rangle$. Recall that we already found that $|E_0\rangle = |0\rangle$ and $|E_1\rangle = e^{i\theta}|1\rangle$ (in some basis), but now we obtain $|E_1\rangle = e^{i\chi}|0\rangle$. Since the two requirements contradict each other, the proof that the above set of states cannot be disentangled (not even to a separable state) is completed. Thus, we have proved that a universal disentangling machine cannot exist. In other words—a single quantum cannot be disentangled.

This result resembles a result regarding two noncommuting mixed states [9]: these states cannot be cloned, and furthermore, they cannot be broadcast.

To summarize, we provided a thorough analysis of disentanglement processes, and we proved that a single quantum cannot be disentangled, that is, the (quantum) nonlocality of an arbitrary state cannot be transformed into classical correlations.

Interestingly, we used a set of four states to prove this, but we conjecture that there are smaller sets that could be used to establish the same conclusion.

The no-cloning of states of composite systems was investigated recently [7,11], and it seems that several interesting connections between these works and the idea of disentanglement can be further explored. For instance, one can probably find systems where the states can only be disentangled (or only be disentangled into product states) if the two subsystems are available together, but *cannot* be disentangled if the subsystems are available one at a time (with similarity to [7]), or *cannot* be disentangled if only bilocal superoperators can be used for the disentanglement process (with similarity to [11]).

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