PHYSICAL REVIEW LETTERS

VOLUME 83 16 AUGUST 1999 NUMBER 7

Exact Solution of Double δ Function Bose Gas through an Interacting Anyon Gas

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(Received 8 December 1998)

A 1D Bose gas interacting through δ, δ' and double δ function potentials is shown to be equivalent to a δ anyon gas, allowing an exact Bethe ansatz solution. In the noninteracting limit, it describes an ideal gas with generalized exclusion statistics and solves some recent controversies.

PACS numbers: 05.30.Jp, 03.70.+k, 11.55.Ds, 71.10.Pm

The concept of particles with generalized exclusion statistics (GES) introduced by Haldane [1] has important consequences [2] in describing 1D non-Fermi liquids [3], which in turn is believed to be related [4] to the edge excitations in fractional quantum Hall effect [5]. On the other hand, inspired by the success of the Chern-Simon theory, an attempt was made recently [6] to describe a 1D ideal gas with GES in the framework of a gauge field model. However, in a subsequent paper [7] the previous result was shown to be wrong and another conclusion was offered. Our aim here is to deal primarily with a 1D Bose gas interacting through double δ function potentials together with the well-known δ and derivative δ-function interactions. We show that this interacting model with several singular potentials is equivalent to a 1D gas with GES (which we call anyon for brevity) interacting via δ-function potential only. This δ-anyon gas is found to be exactly solvable by the coordinate Bethe ansatz (CBA) just as its bosonic counterpart, contradicting the common belief [8] that the CBA is applicable only to models with symmetric or antisymmetric wave functions. Remarkably, at the limit of vanishing interaction, the anyon gas becomes free and gauge equivalent to a related model proposed in [6]. This shows that, though the explicit wave function and the N-body Hamiltonian of [6] are not exact, the conclusion it arrived at is basically correct. Therefore, while the error in the treatment of [6] was detected in [7], the source of this error and the possible way to rectify it becomes evident from our result.

We start with a 1D Bose gas interacting through generalized δ-function potentials as

$$H_N = -\sum_k^N \partial_{x_k}^2 + \sum_{\langle k,l \rangle} \delta(x_k - x_l)[c + i\kappa(\partial_{x_k} + \partial_{x_l})] + \gamma_1 \sum_{\langle j,k,l \rangle} \delta(x_j - x_k)\delta(x_l - x_k) + \gamma_2 \sum_{\langle k,l \rangle} [\delta(x_k - x_l)]^2. \tag{1}$$

This model was briefly considered and readily discarded in [9] as too difficult a problem to solve. Notice, however, that for $\gamma_a = 0, a = 1, 2$, i.e., without the double δ potentials, it has various exactly solvable limits. For example, for $\kappa = 0, c \neq 0$ the model becomes the well-known δ-Bose gas [10], while for $\kappa \neq 0, c = 0$ it corresponds to Bose gas with δ' interaction [11]. Both of these cases are not only exactly solvable by CBA, but also represent quantum integrable systems allowing R-matrix solution. This can be proved through their connection with the quantum integrable nonlinear Schrödinger equation (NLSE) [12] and derivative NLSE [13], respectively. Even the mixed case with $\kappa \neq 0, c \neq 0$ is solvable through CBA [8,11], though as a quantum model it does not allow a R-matrix solution. Nevertheless for $\gamma_a \neq 0$, i.e., with the inclusion of highly singular double δ function interactions, the solvability of the model is completely lost, and the application of the CBA becomes problematic due to the presence of three-body interacting terms. We ask therefore whether, for some choice of the coupling constants γ_a other than zero, this difficulty could still be avoided and the solvability of the model be restored. We find the answer to be affirmative and, in particular, for $\gamma_a = \kappa^2$ the model becomes equivalent to a δ-function anyon gas, which appears to be exactly solvable similar to the well-known bosonic case obtained at $\gamma_a = \kappa = 0$.

Instead of attacking model (1) directly, our strategy would be to transform it into some equivalent tractable problem. For this we notice that, parallel to the relation between the δ-Bose gas and the NLS model [14], the generalized interacting bosonic model (1) can be considered to be the N-particle Hamiltonian of the nonlinear field model:

$$H = \int dx \{:[(\psi_x^\dagger \psi_x + c\rho^2 + i\kappa\rho(\psi^\dagger\psi_x - \psi_x^\dagger\psi)]: + \kappa^2(\psi^\dagger\rho^2\psi)\}, \qquad \rho \equiv (\psi^\dagger\psi), \tag{2}$$

involving bosonic operators $:[\psi(x), \psi^\dagger(y)] = \delta(x-y)$. In (2) we have chosen $\gamma_1 = \gamma_2 = \kappa^2$ and introduced notation : : to indicate normal ordering (NO) in bosonic operators. Restricting now to the $|N\rangle$ particle state and defining the N-particle wave function as

$$\Phi(x_{i_1}, x_{i_2}, \ldots, x_{i_N}) = \langle 0|\psi(x_{i_1})\psi(x_{i_2})\cdots\psi(x_{i_N})|N\rangle, \tag{3}$$

we can generate all terms of (1) starting from (2). For example, the last term in (2): $\int dx\,(\psi^\dagger\rho^2\psi)$ is equivalent to two normally ordered terms like $\int dx :\{\rho^3 + \rho^2\int dy\,[\delta(x-y)]^2\}:$. When the first one acts on the state $|N\rangle$, its three $\psi(x)$ operators in passing through the creation operators at points x_j, x_k, x_l in $|N\rangle$ would produce a sum of terms with a product of three δ functions having arguments $(x - x_j)$, $(x - x_k)$, and $(x - x_l)$. On integration by x, they would generate the double δ function potential $\delta(x_j - x_k)\delta(x_l - x_k)$. Note that this is a three-body term and would not contribute in two-body bosonic interactions. On the other hand, the second term, acting on $|N\rangle$, would give rise to the sum of terms like $\delta(x_k - x_l)\delta(x_k - x_k) \approx [\delta(x_k - x_l)]^2$. Similarly, other terms with δ'- and δ-function interactions are obtained in (1) from (2).

Our next step is to define a gauge transformed operator,

$$\tilde{\psi}(x) = e^{-i\kappa\int_{-\infty}^{x}\psi^\dagger(x')\psi(x')\,dx'}\psi(x), \tag{4}$$

along with its conjugate $\tilde{\psi}^\dagger(x)$. We may check that the derivatives and products of the transformed operators are related to the old ones in the following way:

$$\vdots(\tilde{\psi}^\dagger\tilde{\psi})^2\vdots = :(\psi^\dagger\psi)^2:, \tag{5}$$

$$\begin{aligned} \vdots\tilde{\psi}_x^\dagger\tilde{\psi}_x\vdots &= (\psi_x^\dagger + i\kappa\psi^\dagger\rho)(\psi_x - i\kappa\rho\psi) \\ &= :[\psi_x^\dagger\psi_x + i\kappa\rho(\psi^\dagger\psi_x - \psi_x^\dagger\psi)]: \\ &\quad + \kappa^2(\psi^\dagger\rho^2\psi), \end{aligned} \tag{6}$$

where $\vdots\;\vdots$ stands for NO with respect to the transformed operator (4), which does not necessarily coincide with the bosonic NO as evident from (6). Using these relations, therefore, one rewrites Hamiltonian (2) in the form

$$\tilde{H} = \int dx \vdots[\tilde{\psi}_x^\dagger\tilde{\psi}_x + c(\tilde{\psi}^\dagger\tilde{\psi})^2]\vdots. \tag{7}$$

Note, however, that, in spite of the same form as NLSE, (7) is not the same as the known model, since the fields involved are no longer bosonic operators but exhibit *anyon*like properties

$$\begin{aligned} \tilde{\psi}^\dagger(x_1)\tilde{\psi}^\dagger(x_2) &= e^{i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}^\dagger(x_1), \\ \tilde{\psi}(x_1)\tilde{\psi}^\dagger(x_2) &= e^{-i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}(x_1) + \delta(x_1 - x_2), \end{aligned} \tag{8}$$

etc., where

$$\epsilon(x - y) = \pm 1 \quad \text{for } x > y, x < y, \text{ and } = 0 \text{ for } x = y, \tag{9}$$

[which may be expressed also through the symmetrical unit-step function [15]. This means that the bosonic commutation relation (CR) $[\tilde{\psi}(x), \tilde{\psi}^\dagger(y)] = \delta(x-y)$ remains valid at the coinciding points. These relations can be checked easily by using realization (4) through bosonic fields.

For finding an N-body Hamiltonian corresponding to (7), we observe that operator $\tilde{\psi}(x)$ in passing through the string of anyonic creation operators in $|\tilde{N}\rangle$ would pick up first a phase $e^{-i\kappa\sum_{i<k}\epsilon(x-x_i)}$ due to (8) and then leave a $\delta(x - x_k)$ at x_k due to its standard CR at the coinciding points. The phase factor, however, is canceled subsequently when the associated $\tilde{\psi}^\dagger(x)$ also passes through the same creation operators and comes to the point x_k. This happens due to the opposite signs of the phases as seen from (8). Therefore, finally, similar to the bosonic model one obtains a δ-function interacting gas:

$$\tilde{H}_N = -\sum_k^N \partial_{x_k}^2 + c\sum_{\langle k,l\rangle}\delta(x_k - x_l). \tag{10}$$

However, in contrast to the standard case, the wave function now exhibits a generalized symmetry:

$$\tilde{\Phi}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = e^{-i\kappa\left[\sum_{k=i+1}^{j}\epsilon(x_i-x_k) - \sum_{k=i+1}^{j-1}\epsilon(x_j-x_k)\right]}\tilde{\Phi}(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N), \tag{11}$$

dictated by the operator relations (8), and the model is defined in an infinite-interval space R. It should be emphasized that, due to the validity of bosonic CR at coinciding points in the anyonic relations (8), in the induced wave function (11) the phase factor with $\epsilon(x_l - x_k)$ vanishes at $x_l = x_k$ making it well defined. Note that the commutation relations (8) and the symmetry of the wave function (11) for this 1D model are remarkably consistent with the genuine anyonic behaviors in 2D [16]. Like the anyonic property, while $\tilde{\psi}^\dagger$ is $q^{-1} = e^{i\kappa\epsilon(x_1-x_2)}$ symmetric, the wave function $\tilde{\Phi}(x_1, x_2)$ is q symmetric, so that the $|\tilde{2}\rangle$-particle state remains invariant under permutation of coordinates. It is evident that the range of κ is sufficient to be restricted in the interval of 2π by choosing $-\pi \geq \kappa \leq \pi$ and any nonzero value of κ affects the symmetry under reflection of coordinates, close to the 2D anyon case. Another parallel of multianyon wave function [16] can be observed in (11); i.e., the phase factor appearing under exchange of two of its arguments also depends on the intermediate coordinates.

We have converted thus the original eigenvalue problem related to (1) to an equivalent one: $\tilde{H}_N\tilde{\Phi}(x_1, \ldots, x_N) = E_N\tilde{\Phi}(x_1, \ldots, x_N)$ for (10) acting on an anyon-type wave function (11). Since the Bethe ansatz solution is meant apparently for the (anti)-symmetric wave functions only [8], the present problem claims novelty. However, we find that the CBA is applicable here with equal success, if one modifies the Bethe ansatz for the wave function appropriately:

$$\tilde{\Phi}(x_1, \ldots, x_N) = \Phi_A(x_1, \ldots, x_N)\Phi_B(x_1, \ldots, x_N). \quad (12)$$

Here, Φ_B is the symmetric function in the standard Bethe ansatz form [10]:

$$\Phi_B(x_1, \ldots, x_N) = \sum_P A(P)e^{i\sum_j x_j k_{P_j}}, \quad (13)$$

defined in the primary region R_1: $x_1 \leq x_2 \leq \cdots \leq x_N$, while Φ_A is the additional anyonic part given as

$$\Phi_A(x_1, \ldots, x_N) = e^{-i(\kappa/2)\Lambda(x_1,\ldots,x_N)}, \quad \text{with } \Lambda \equiv \sum_{i<j}\epsilon(x_i - x_j), \quad (14)$$

with $\epsilon(x_i - x_j)$ as defined in (9). Remarkably, the discontinuity in the derivative of the wave function (12) at the boundary,

$$(\partial_{x_l} - \partial_{x_k})\tilde{\Phi}|_+ - (\partial_{x_l} - \partial_{x_k})\tilde{\Phi}|_- = c\tilde{\Phi}|_0, \quad (15)$$

with the notation $|_\pm = |_{x_l = x_k^\pm}$ and $|_0 = |_{x_l = x_k}$, determines the scattering amplitude in the same way as in the case of δ-Bose gas. This becomes possible since using (12) and (14), along with (9), one obtains

$$\tilde{\Phi}(x_1, \ldots, x_N)|_+ = e^{-i(\kappa/2)(-1+S)}\Phi_B(x_1, \ldots, x_N)|_{x_l = x_k^+},$$
$$\tilde{\Phi}(x_1, \ldots, x_N)|_- = e^{-i(\kappa/2)(+1+S)}\Phi_B(x_1, \ldots, x_N)|_{x_l = x_k^-},$$
$$\tilde{\Phi}(x_1, \ldots, x_N)|_0 = e^{-i(\kappa/2)(S)}\Phi_B(x_1, \ldots, x_N)|_{x_l = x_k}, \quad (16)$$

where $S = \Lambda - \epsilon(x_k - x_l)$. Contributions coming from the derivatives of other $\epsilon(x_i - x_j)$ factors (as δ functions) in (15) clearly cancel each other, transforming it, consequently, to an equation only for the symmetric part of the wave function in the standard form:

$$(\partial_{x_l} - \partial_{x_k})\Phi_B(x_1, \ldots, x_N)|_+ = \tilde{c}\Phi_B(x_1, \ldots, x_N)|_0, \quad (17)$$

but with modified coupling constant $\tilde{c} = c/(4\cos\frac{\kappa}{2})$. Note that the singularity of the original bosonic problem is reflected in the discontinuity of the anyonic wave function (12). Such discontinuity at the boundaries of different regions, though somewhat unusual in CBA, has been observed recently in another context [17]. In the present case, this also does not affect the physical picture, as evident from the reduction of the problem to (17) for continuous wave functions. Therefore following arguments of the bosonic model [10], one can reduce (17) further to the region R_1 involving only adjacent k's; i.e., with $x_l \to x_{k+1}$ and using the Bethe ansatz (13), calculate the two-particle scattering amplitude as $e^{i\theta_{ll+1}} = (k_l - k_{l+1} - i\frac{\tilde{c}}{2})/(k_l - k_{l+1} + i\frac{\tilde{c}}{2}) = e^{-i\theta_{l+1l}}$. Notice, however, that in contrast to [10] the coupling constant in this anyonic case has been changed. At $\kappa = 0$ one recovers the bosonic case, while $\kappa = \pm\pi$ gives hard-core repulsion $c \to \infty$, simulating a fermionic model. In the present infinite-interval space, the values of $\{k\}$ have no restriction. However, if we restrict to the interval $0 \leq x_j \leq L$, the boundary condition would be twisted as $\tilde{\Phi}(x_{1+L}, \ldots, x_N) = e^{i\kappa(N-1)}\tilde{\Phi}(x_1, \ldots, x_N)$, and one would obtain the determining equations for k's as $k_j = -\frac{1}{L}\sum_{s=1}^N \tilde{\theta}_{js} + \frac{2\pi}{L}n_j + \frac{\kappa}{L}(N + 1 - 2j)$, $j = 1, \ldots, N$, with n_j an integer. Here we have redefined $\tilde{\theta}_{js} = \theta_{js} - \kappa\epsilon(j - s) = -\tilde{\theta}_{sj}$ to introduce an anyonic scattering phase, since such a particle in passing through others would pick up a phase $e^{\pm i\kappa}$ depending on its relative position. The energy eigenvalue of the system $E_N = \sum_j^N k_j^2$, though it has the same form as in the bosonic model, acquires in effect a different value due to the changed coupling constant. We may mention here that the possibility of gauging away certain multispecies fermionic interaction by introducing a twisting in the boundary condition has been observed recently [18]. Similar spirit of the present result in a totally different context might therefore be an indication of a deeper generality.

Now at $c \to 0$ limit of (10) we recover the results related to a GES ideal gas having properties similar to (8) and (11). As we have shown, this free anyonic Hamiltonian would be equivalent to the N-bosonic model (1) at $c = 0, \gamma_a = \kappa^2$. Establishing such a fact through direct gauge transformation of the wave function was the aim of [6]. However, this seems to be difficult to achieve, in which investigations [6,7] were concentrated. We have avoided this difficulty by showing the equivalence through the gauge relation between the related field models (7) and (2). Notice also that the field model [Eq. (8)] of [6] differs from that [Eq. (6)] of [7] and ours by a crucial term. We note again that (1) even with $c = 0$ involves three-body terms, ignoring which naturally would make it not equivalent to the free anyonic Hamiltonian. In both [6] and [7], the authors worked with a two-body bosonic Hamiltonian which could not give the right equivalence.

The author thanks Partha Mitra for helpful discussions and Diptiman Sen for providing useful references.

Support of the AvH Foundation, Germany, is gratefully acknowledged.

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