Time-Dependent Ginzburg–Landau Analysis of Inhomogeneous Normal-Superfluid Transitions

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We study the propagation of the normal-superfluid interface under inhomogeneous cooling. Assuming a uniform temperature gradient we establish the conditions for creating topological defects for both slow and fast superfluid transitions using the time-dependent Ginzburg-Landau theory. For fast transitions, we find agreement with the Kibble-Zurek scenario. Experiments where the temperature change is generated by absorption of a neutron in ^{3}He are discussed.

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Superfluid ^{3}He can be heated locally above the superfluid transition temperature T_c by absorption of a neutron [1]. The number of vortices [2] and the thermal energy [3] generated in this process were measured in recent experiments. It was argued that vortices and other defects are created when the heated region cools back to the superfluid state. The results of the experiments were interpreted as evidence of the Kibble-Zurek mechanism of defect formation at a rapid phase transition [4].

The purpose of this Letter is a more detailed analysis of the defect formation near a propagating second-order phase interface. The original Kibble-Zurek estimate applies to a spatially constant temperature $T(\mathbf{r}, t) \equiv T(t)$ which cannot be realized in a condensed-matter system. These ideas were generalized to inhomogeneous temperature distribution by Kibble and Volovik [5]. While our work was in progress, we learned about the numerical work in Ref. [6]. An important point not discussed in these papers is that the creation of defects depends on the *amplitude* of fluctuations in the system.

The energy released in the nuclear reaction ($n + {}^3\mathrm{He} \rightarrow {}^3\mathrm{H} + p$) is 764 keV. This should be compared to the energy of an elementary vortex ring created during the cooldown which is on the order of 10 meV. We are not looking for a theory that could give a detailed description of the energy flow through the interval of 7 orders of magnitude. Instead, we consider the simplest but realistic model. In particular, we assume that the temperature is well defined, and use the time-dependent Ginzburg-Landau (TDGL) theory to determine the dynamics of the superfluid order parameter. We derive the conditions for defect formation in both rapid and slow cooling. During a rapid cooling, a large number of vortices is initially created in a supercooled region according to the Kibble-Zurek scenario, and these survive the interaction with the normal-superfluid (N-S) interface propagating from the bulk liquid. In a slow cooling, defects are created by sufficiently strong fluctuations beyond the N-S interface. The theory can be used for superconductors and superfluid ^{3}He. Being applied to the neutron experiments in ^{3}He, it confirms the scenario of fast phase transitions.

Propagation of the N-S interface.—We assume a temperature distribution $T(\mathbf{r}, t)$ that has gradient in the x direction. The length scale of the gradient is $\lambda = T_c(\partial T/\partial x)^{-1}$ and the time scale of cooling $\tau_Q = -T_c(\partial T/\partial t)^{-1}$. The derivatives are taken at the point where $T = T_c$. The temperature profile moves along the x axis with the velocity $v = \lambda/\tau_Q$.

Our analysis is based on the TDGL model for a scalar order parameter. We thus neglect the complicated structure of defects which may exist in real ^{3}He superfluid. As is well known, the TDGL model can be applied only for gapless systems (see Ref. [7] for a review). For superfluid ^{3}He, this restricts the temperature to a very narrow vicinity of T_c such that $\tau\Delta(T) \ll \hbar$. Here τ is the mean free time of quasiparticles and Δ the pairing amplitude. Therefore, our analysis is valid only for a very initial stage of the development of a superfluid phase where the characteristic time scale for the variation of the order parameter is on the order of $\hbar/T_c$. At later stages, when $\Delta(T)$ exceeds $\hbar/\tau$, the speed of variations of Δ decreases and the characteristic time becomes comparable with τ.

We start with the TDGL equation

$$\tau_0 \frac{\partial \Delta}{\partial t} = -\alpha\Delta - \beta|\Delta|^2\Delta + \gamma \frac{\partial^2 \Delta}{\partial x^2} \tag{1}$$

with the standard notations $\tau_0 = \pi\hbar/8T_c$, $\alpha = T/T_c - 1$, $\beta = 7\zeta(3)/8\pi^2T_c^2$, and $\gamma = 7\zeta(3)\xi_0^2/12$. Here $\xi_0 = \hbar v_F/2\pi T_c$ is the superfluid coherence length and v_F is the Fermi velocity. A propagating superfluid-normal interface has $\alpha(x,t) = \alpha(x - vt) = (d\alpha/dx)(x - vt)$. The point $x = vt$ corresponds to $T = T_c$. For $(d\alpha/dx) > 0$, the cool part is to the left, while the temperature front moves towards the hot part at the right. We introduce dimensionless variables z, $\tilde{t}$, and Ψ:

$$x = z\xi_0\left(\frac{7\zeta(3)}{12}\frac{\lambda}{\xi_0}\right)^{1/3}; \qquad t = \tilde{t}\tau_0\left(\frac{12}{7\zeta(3)}\frac{\lambda^2}{\xi_0^2}\right)^{1/3};$$
$$\Delta = \Psi\sqrt{\frac{2}{3}}\,\pi T_c\left(\frac{12}{7\zeta(3)}\frac{\xi_0}{\lambda}\right)^{1/3}. \tag{2}$$

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The velocity of the temperature front $v = \lambda/\tau_Q$ in dimensionless form is then

$$u = \left(\frac{3\pi^3}{14\zeta(3)}\right)^{2/3}\left(\frac{\lambda}{\xi_0}\right)^{1/3}\frac{v}{v_F} \sim \left(\frac{\lambda}{\xi_0}\right)^{4/3}\frac{\tau_0}{\tau_Q}. \quad (3)$$

We look for a solution which propagates with the same velocity as the temperature front: $\Psi(z,\tilde{t}) = \Psi(z - u\tilde{t})$. Equation (1) reduces to

$$\frac{d^2\Psi}{dz^2} + u\frac{d\Psi}{dz} - z\Psi - \Psi^3 = 0. \quad (4)$$

The boundary conditions are $\Psi \to 0$ for $z \to +\infty$ and $\Psi \to \sqrt{-z}$ for $z \to -\infty$. We assume a real Ψ. It implies that we neglect the supercurrent which is induced by the temperature gradient. This is justified in Fermi liquids because the conversion of the normal heat conduction into superfluid counterflow occurs at distances much longer than the characteristic width of the N-S interface [8]. In ^{3}He this limits us to $z \gg -T_F/T_c$, where $T_F \sim 10^3 T_c$ is the Fermi temperature. The linear temperature profile will cease to hold for $z \lesssim -T_F/T_c$.

We solve Eq. (4) numerically. The results are presented in Fig. 1 for different values of u. The solution exists for all values of u. As the velocity increases the superfluid domain lags behind the temperature front by the distance z_0, which is approximately equal to $u^2/4$. Equivalently, the time lag $\tilde{t}_0 = z_0/u \approx u/4$.

Rapid cooling.—The results in Fig. 1 can be understood using a universal solution of Eq. (4) which is available in the limit $u \gg 1$. There are three overlapping regions with different behaviors of Ψ as a function of z: (i) In the region where Ψ is small, we can neglect the cubic term and write $\Psi = \chi\exp(-uz/2)$. We get

$$\frac{d^2\chi}{dz^2} - \left(z + \frac{u^2}{4}\right)\chi = 0. \quad (5)$$

The solution for χ is the Airy function shifted towards negative z by $z_0 = u^2/4$. It decays exponentially with increasing z in the region $z > -z_0$. (ii) The nonlinearity of the full equation (4) becomes important around the maximum of the Airy function located at $z \approx -z_0$, where a rapid growth of the order parameter forms the N-S interface. The position of the interface on the temperature profile determines the local temperature and thus both the order parameter magnitude and the spatial dimension of the interface. It is convenient to use the local coordinate $\tilde{z}$ and $\tilde{\Psi}$ according to $z = -z_0 + \tilde{z}/\sqrt{z_0}$ and $\Psi = \sqrt{z_0}\,\tilde{\Psi}$. Equation (4) takes the form

$$\frac{d^2\tilde{\Psi}}{d\tilde{z}^2} + \frac{u}{z_0^{1/2}}\frac{d\tilde{\Psi}}{d\tilde{z}} + \tilde{\Psi} - \tilde{\Psi}^3 - \frac{\tilde{z}}{z_0^{3/2}}\tilde{\Psi} = 0. \quad (6)$$

For $z_0 = u^2/4$, the last term is small while the first four terms constitute a parameter-free equation since the coefficient of the second term equals 2. They determine the fast increase of Ψ up to its local equilibrium value $\Psi \approx \sqrt{z_0}$. The interface width in $\tilde{z}$ coordinates is on the order of unity. (iii) In the superfluid region, $\tilde{z} \ll -1$, the solution has the asymptotics $\Psi = (-z)^{1/2} - (u/4)(-z)^{-3/2} + O(z^{-5/2})$.

A large supercooled normal region is unstable towards fluctuations of the order parameter. Their growth is determined by the linearized version of Eq. (1). Neglecting the spatial inhomogeneity, it reduces to

$$\frac{\partial\Psi}{\partial\tilde{t}} = u\tilde{t}\Psi. \quad (7)$$

This has the solution $\Psi(\tilde{t}) = \Psi(0)\exp(u\tilde{t}^2/2)$ with the time constant $\tilde{\tau} = \sqrt{2/u}$. In the case of a constant temperature, the dominant thermal fluctuations have [9]

$$\langle|\Delta - \Delta_{\rm eq}|^2\rangle \sim \frac{T_c}{\alpha N(0)\xi^3(T)}, \quad (8)$$

where $\xi(T) = \sqrt{\gamma/\alpha}$ and $N(0)$ is the density of quasiparticle states at the Fermi surface. According to Zurek, the dominant fluctuations in a temperature sweep correspond to the temperature T_Z which is reached in time $\tilde{\tau}$, i.e., $1 - T_Z/T_c \sim \sqrt{\tau_0/\tau_Q}$ [4]. The length scale of such fluctuations $\tilde{\xi} \sim (u\tilde{\tau})^{-1/2} \sim u^{-1/4}$ is fully determined by u. However, the dimensionless initial amplitude

$$\langle\Psi^2(0)\rangle \sim \frac{\sqrt{u}}{N(0)T_c\xi_0^3}\left(\frac{\tau_Q}{\tau_0}\right)^{1/4} \quad (9)$$

is an independent parameter in the theory. Equation (9) corresponds to a lower limit of fluctuations because experiments may contain also other fluctuations arising, for example, from nonmonotonous temperature distribution as in the "baked Alaska" scenario [10]. For simplicity, we assume that only thermal fluctuations are present.

For large u, the size of a superfluid nucleus is small compared to the width of the supercooled region: $z_0/\tilde{\xi} \sim u^{9/4}$. The fluctuations grow initially according to Eq. (7). Since the time needed for the interface to propagate through the supercooled region $\tilde{t}_0 = u/4$ is large compared to $\tilde{\tau}$ the fluctuations will have time to grow by a factor $\exp(u^3/32) \gg 1$. The phases of Ψ in different nuclei

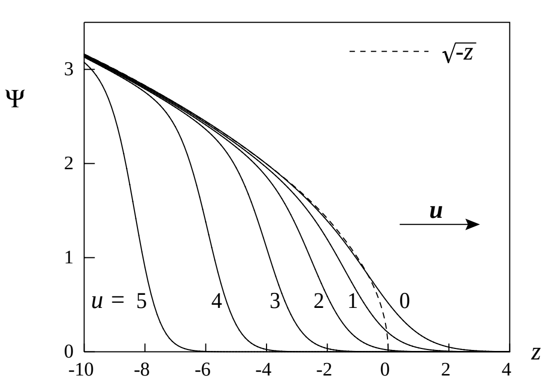

FIG. 1. The order parameter front [solution of Eq. (4)] for different values of the dimensionless velocity u. The dashed line gives the solution ignoring all derivative terms in (4).

are uncorrelated. After coalescense, the random phases produce topological defects such as vortices, according to the Kibble-Zurek scenario [4]. Ultimately the superfluid nuclei in the supercooled region meet the propagating N-S interface. This collision leads to a profound perturbation of the interface if $|\Psi|$ in a nucleus has the same order of magnitude as that at the N-S interface. Using (9), the condition for this instability is

$$u^{-6}\exp(u^3/4) \gtrsim [N(0)T_c\xi_0^3]^4(\tau_0/\tau_Q) \quad \text{for } u \gg 1\,. \tag{10}$$

Defects in bulk superfluid.—The next step is to study how the perturbations of the interface can enter the bulk superfluid, i.e., to study how vortices or other defects separate from the interface. We illustrate this process with the one-dimensional model used above. The conclusions are qualitatively the same as we can anticipate for vortices near a moving N-S interface.

Defects in 1D should be modeled by a real Ψ. Consider a nucleus with a negative $\Psi(z)$ located on the right hand side of the interface shown in Fig. 1. If the nucleus is strong enough, it can create a kink in the order parameter such that $\Psi(z)$ changes its sign and then continues as negative valued; see Fig. 2. The stability of the kink on the interface can be studied using Eq. (6), where $-z_0$ now denotes the location of the kink. Assume first $z_0 \gg 1$ and $u \ll 1$. In this case, we can treat the second and last terms as small perturbations. The unperturbed equation has the solution $\tilde{\Psi}_0(\tilde{z}) = \pm\tanh(\tilde{z}/\sqrt{2})$. Writing $\tilde{\Psi} = \tilde{\Psi}_0 + \tilde{\Psi}_1$ in Eq. (6), we find for the correction $\tilde{\Psi}_1$

$$\frac{d^2\tilde{\Psi}_1}{d\tilde{z}^2} + \tilde{\Psi}_1 - 3\tilde{\Psi}_0^2\tilde{\Psi}_1 = \frac{\tilde{z}}{z_0^{3/2}}\tilde{\Psi}_0 - \frac{u}{z_0^{1/2}}\frac{d\tilde{\Psi}_0}{d\tilde{z}}\,. \tag{11}$$

It has a solution only for certain z_0. The solvability condition is found by multiplying Eq. (11) with $d\tilde{\Psi}_0/d\tilde{z}$ and integrating over the kink. The left hand side vanishes while the right hand side implies $z_0 = 3/2u$. This defines an unstable equilibrium position for the kink. The numerical solution of Eq. (1) shows that, if the kink is located to the right of $-z_0$, i.e., $-3/2u < z_{\mathrm{kink}}$, it will move with a velocity larger than u, and thus will be absorbed to the interface. On the contrary, it will remain in the superfluid if placed to the left, $z_{\mathrm{kink}} < -3/2u$, because its velocity is smaller than u. We see that with increasing u, the critical position approaches the interface and cannot be separated from it any more when $u \sim 1$. It means that the defects created under condition (10) in the supercooled region will remain in the superfluid after the passage of the interface.

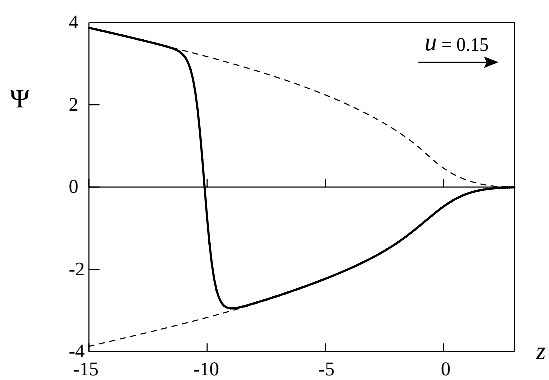

FIG. 2. A kink in the order parameter near its critical location $z \approx -3/2u$, where the whole profile moves with the same velocity.

There is no supercooled region in a slow transition, $u \ll 1$. Nevertheless, defects can be created directly within the growing bulk superfluid if the amplitude of the fluctuations (8) is larger than the equilibrium order parameter at the critical position of the kink:

$$u^3 \gtrsim [N(0)T_c\xi_0^3]^4(\tau_0/\tau_Q) \quad \text{for } u \ll 1\,. \tag{12}$$

Discussion.—Equations (10) and (12) constitute the conditions for creating topological defects at an inhomogeneous phase transition. There is no universal critical value of u; it depends on the fluctuations. The conditions in fast ($u \gg 1$) and slow ($u \ll 1$) transitions approach a common limit when $u \to 1$. This point corresponds to $\tau_Q/\tau_0 \approx [N(0)T_c\xi_0^3]^4$ and $\lambda/\xi_0 \approx [N(0)T_c\xi_0^3]^3$ such that the Zurek temperature T_Z equals the Ginzburg temperature T_G where the fluctuations are comparable to the mean-field order parameter, $1 - T_G/T_c \sim [N(0)T_c\xi_0^3]^{-2}$. The parameter $N(0)T_c\xi_0^3 \sim T_F^2/T_c^2$ is large in ^{3}He.

The slow transition result (12) is, of course, on the borderline of applicability of the GL theory. Nevertheless it shows qualitatively where the mean-field picture of the moving interface breaks down and the defects can be spontaneously created. Equation (12) can be compared to experiments where different types of vortices of ^{3}He-A were detected in a rotating container after a usual cooldown (no nuclear reactions) [11]. The relative abundance of two vortex types was found to depend on the cooling rate. A changeover was seen at $\tau_Q = 3000$ s. Assuming that this value corresponds to the instability condition (12), we can translate it to a critical $u \sim 0.04$ and $\lambda \sim 10$ m. The experimental parameters are not well known but they agree in order of magnitude with these values.

If $[N(0)T_c\xi_0^3]^4(\tau_0/\tau_Q) = (T_c - T_Z)^2/(T_c - T_G)^2$ is large, the fluctuations are relatively weak and a large supercooled region is required for creation of defects. Equation (10) determines the condition where the cooling can be regarded as a rapid quench for which the Kibble-Zurek scenario of vortex formation works. Defects can always be formed at sufficiently large u because the left hand side of Eq. (10) grows exponentially with u. The condition (10) ceases to be accurate for very high cooling rates ($\tau_Q \lesssim \tau_0[N(0)T_c\xi_0^3]^2$) where the superfluid front gets outside of the validity region of the TDGL theory.

The condition (10) can be fulfilled for neutron triggered phase transitions in ^{3}He. Let us consider this case in more detail. If the temperature profile is determined by heat conduction we have

$$T(r,t) = T_0 + \frac{E_0}{C}\frac{\exp(-r^2/4Dt)}{(4\pi Dt)^{3/2}}, \tag{13}$$

where $T_0 < T_c$ is the initial temperature. The specific heat C and the diffusion coefficient $D = v_F \ell/3$ are assumed constant, $\ell = v_F \tau$ being the quasiparticle mean free path. We assume that the deposited energy E_0 is large enough to heat the volume of the order of ℓ^3 to a temperature much higher than the critical temperature: $E_0 \gg \ell^3 C(T_c - T_0)$. The time

$$t_{\max} = \frac{1}{4\pi D}\left[\frac{E_0}{C(T_c - T_0)}\right]^{2/3} \gg \tau \tag{14}$$

is needed for the temperature at $r = 0$ to decrease down to T_c. Equation (13) gives

$$\lambda = \frac{T_c}{T_c - T_0}\sqrt{\frac{2Dt}{3\ln(t_{\max}/t)}}, \tag{15}$$

$$\tau_Q = \frac{2T_c t}{3(T_c - T_0)}[1 - \ln(t_{\max}/t)]^{-1} \tag{16}$$

so that λ is much longer than ℓ and the cooling rate is $\tau_Q \gtrsim t_{\max}$. The temperature front defined by $T(r_0) = T_c$ has the radius $r_0^2 = 6Dt\ln(t_{\max}/t)$. The front starts to move towards the center of the hot bubble with the velocity $v = \lambda/\tau_Q$ for $t > t_{\max}/e$. The velocity parameter $u^3 \sim [T_c/(T_c - T_0)](\tau/t_{\max})(\ell/\xi_0)$.

With the experimental values $\lambda \approx 10^{-2}$ cm and $\xi_0 \approx 10^{-6}$–10^{-5} cm, we get $(\xi_0/\lambda)^{1/3} \approx 10^{-1}$. The front velocity is roughly $v \sim 10^{-1} v_F$. This estimate suggests that $u \sim 1$, i.e., at the initial moment, the velocity of the temperature front u is small and the superfluid domain expands without delay into the cooled region. However, when t approaches $t_{\max}$, both the scale λ and the front velocity $v = \lambda/\tau_Q$ increase according to Eqs. (15) and (16). The velocity parameter u in Eq. (3) increases as $(t_{\max} - t)^{-2/3}$ and reaches $u_{\max} \sim u_0(t_{\max}/\tau)^{2/3}$ before the divergence is cut off at $t_{\max} - t \sim \tau$. Under the conditions of experiment [2], $u_{\max}$ can be as high as 10–100. We conclude that the regime of cooling by diffusion, at least in its final stage, corresponds to a rapid quench: the value $u \gtrsim 4$ needed to satisfy Eq. (10) is well within the experimental range.

The instability inside a big supercooled volume creates roughly $N_0 \sim (4\pi/3)[r_0/\xi(T)]^3$ initial topological defects (vortices) through the Kibble-Zurek mechanism. The radius r_0 should be taken at the moment when u reaches its critical magnitude $u \gtrsim 4$ such that the instability occurs. We estimate $N_0 \sim (\ell/\xi_0)^3 u^{-9/4}$. This number is large due to a big ratio ℓ/ξ_0 for ^{3}He. For large u, almost all of the initial vortices remain in the superfluid after the passage of the N-S interface. Many of the small vortex rings shrink away, but some may grow in the velocity field created by other vortices and by external conditions (rotation of the vessel). Finally some vortices are detected in the experiments of Ref. 2. However, it is difficult to say definitely that the observed vortices are created by the Kibble-Zurek mechanism: vortices might also nucleate when the hot bubble distorts the externally applied flow field and also nonthermal fluctuations may contribute to vortex nucleation.

The possibility to realize the Kibble-Zurek scenario via diffusive cooling depends crucially on the ratio of the mean free path and the zero-temperature coherence length: large $u \gg 1$ can be achieved only for systems with $\ell \gg \xi_0$. Such systems are, for example, superfluid ^{3}He and clean superconductors. On the contrary, if $\ell \lesssim \xi_0$, as is the case in dirty superconductors, the velocity parameter $u \lesssim 1$, and defects can be created only by strong fluctuations under conditions of Eq. (12).

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