Optical Solitary Wave and Shock Solutions of the Higher Order Nonlinear Schrödinger Equation

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An exact analytical solution describing the evolution of nonlinear optical envelope pulses in an optical fiber is obtained. The propagation of the pulse is modeled by a higher order nonlinear Schrödinger equation containing third order and nonlinear dispersive terms. This solution includes pulses that are symmetric solitary waves, asymmetric solitary waves, and asymmetric structures that develop into an optical shock. [S0031-9007(98)08242-8]

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Envelope pulses propagating in nonlinear dispersive media experience group velocity dispersion and nonlinear effects such as self-phase modulation. In the anomalous dispersive regime these are competing effects, and when these two effects balance, the initial pulse can evolve into a soliton or solitary wave. In particular, monomode optical fibers support soliton propagation owing to an intensity-dependent index of refraction. The propagation of these picosecond optical pulses is modeled [1] by the nonlinear Schrödinger (NLS) equation, which is integrable [2] by the inverse scattering method yielding multisoliton solutions. The validity of the NLS equation as a reliable model is dependent on the assumption that the spacial width of the soliton is much larger than the carrier wavelength. This is equivalent to the condition that the width of the soliton frequency spectrum is much less than the carrier frequency, $\delta\omega \ll \omega_0$. The robustness of the optical soliton makes it useful for long distance optical communication systems, the high frequency of the optical carrier makes possible a high bit rate, and to increase the bit rate further it is desirable to use shorter femtosecond pulses. In order to model these shorter pulses additional terms such as third order and nonlinear dispersion have been included [3] in the NLS equation, resulting in the so-called higher order nonlinear Schrödinger (HONLS) equation. Many authors have relied on the inclusion of extra terms to model ultrashort pulses, but recently it was proven [4,5] that these pulses cannot be modeled by simply including higher order terms in the NLS equation. Generally, the NLS model of any order will break down [6] if $\delta\omega < \omega_0$. Nevertheless, it is still useful to study the HONLS equation since it does model optical solitons in fibers when the condition $\delta\omega \ll \omega_0$ is satisfied.

The form of the HONLS equation has been previously derived [7–9] for monomode optical fibers

$$i\varphi_\zeta + \tfrac{1}{2}\varphi_{\tau\tau} - N^2|\varphi|^2\varphi + i\varepsilon\varphi_{\tau\tau\tau} - iN^2[\beta_1|\varphi|^2\varphi_\tau + \beta_2\varphi(|\varphi|^2)_\tau] = 0\,, \quad (1)$$

where the independent variables are related to the fiber coordinate z and the time t by

$$\zeta = \frac{|D_2|z}{T_0^2}\,, \qquad \tau = \frac{t - z/v_g}{T_0}\,. \quad (2)$$

In the above equations D_2 is the group velocity dispersion coefficient, T_0 is the pulse width, and v_g is the group velocity. The other coefficients are related to the fiber parameters by

$$N^2 = \frac{n_2\omega_0 P_0 T_0^2}{cA|D_2|}\,, \qquad \varepsilon = \frac{|D_3|}{6|D_2|T_0}\,,$$
$$\beta_1 = \frac{2}{\omega_0 T_0} + \frac{n'}{nT_0} + \frac{2r'}{rT_0}\,, \qquad \beta_2 = \beta_1 + \frac{2r'}{rT_0}\,, \quad (3)$$

where D_3 is the third order dispersion coefficient, n is the linear index of refraction, n_2 is the Kerr coefficient, c is the speed of light, A is the effective core area, P_0 is the peak input power, and r is the frequency-dependent core radius. The primes indicate differentiation with respect to frequency and all frequency-dependent parameters are evaluated at the carrier frequency ω_0. The first three terms of Eq. (1) comprise the NLS equation, the fourth term is the third order dispersive term, and the last two terms are generalized self-steepening terms resulting from an intensity dependence of the group velocity. From Eq. (3) it is obvious that these last three terms become more important as the pulse width decreases. In this Letter we obtain an exact analytical solution of Eq. (1) which includes both solitary wave solutions and shock solutions for arbitrary values of the fiber parameters.

Previous analytical solution of the HONLS equation began with the special case when $\varepsilon = 0$. Kaup and Newell [9] have shown that when $D_2 = 0$ Eq. (1) is integrable by the inverse scattering method leading to soliton solutions. In the more general case of arbitrary D_2 exact solitary wave solutions [10–12] were obtained, and pulse distortion leading to shock formation [13–16] was also studied. These analytical solutions exhibit an interesting difference from the NLS soliton in that there is an intensity-dependent carrier wave phase shift across the pulse that is not a property of the NLS soliton. When a nonzero ε is included in Eq. (1) exact solitary wave solutions [8,17–20] have been obtained for special values of the parameters, and there are two recent articles [21,22] giving solitary wave solutions for arbitrary values of parameters in the HONLS equation.

In order to obtain both solitary wave and shock solutions we proceed as in [16] and express the complex envelope function $\varphi(\zeta,\tau)$ as

$$\varphi(\zeta,\tau) = u(\zeta,\tau)e^{i(\kappa\zeta-\Omega\tau)+i\sigma(\zeta,\tau)}, \tag{4}$$

where $u(\zeta,\tau)$ and $\sigma(\zeta,\tau)$ are real amplitude and phase functions, respectively, with κ and Ω being small deviations from the carrier wave number and frequency. Next, Eq. (1) is split into its real and imaginary parts

$$\tfrac{1}{2}(1+6\varepsilon\Omega)u_{\tau\tau} - u\sigma_\zeta - (\kappa + \tfrac{1}{2}\Omega^2 + \varepsilon\Omega^3 - \Omega\sigma_\tau - 3\varepsilon\Omega^2\sigma_\tau)u - \tfrac{1}{2}(1+6\varepsilon\Omega)\sigma_\tau^2 u - N^2(1+\beta_1\Omega - \beta_1\sigma_\tau)u^3 - \varepsilon(u\sigma_{\tau\tau\tau} + 3\sigma_{\tau\tau}u_\tau + 3\sigma_\tau u_{\tau\tau} - \sigma_\tau^3 u) = 0, \tag{5}$$

$$\varepsilon u_{\tau\tau\tau} + u_\zeta + \tfrac{1}{2}(1+6\varepsilon\Omega)(u\sigma_{\tau\tau} + 2\sigma_\tau u_\tau) - (\Omega + 3\varepsilon\Omega^2)u_\tau - N^2(\beta_1 + 2\beta_2)u^2u_\tau - 3\varepsilon(u\sigma_{\tau\tau}\sigma_\tau + \sigma_\tau^2 u_\tau) = 0. \tag{6}$$

The first step in the solution of these two equations is use of the ansatz

$$\sigma_\tau = Au \tag{7}$$

to eliminate the dependent variable σ resulting in an overdetermined system. In order for these two equations to be compatible we use the next ansatz

$$2uu_{\tau\tau} - u_\tau^2 = g(u), \tag{8}$$

where g is a function of u that will be determined later. Using these, Eq. (6) can be integrated and expressed in the form of a nonlinear wave equation

$$(u^2)_\zeta + v(u)(u^2)_\tau = 0, \tag{9}$$

which has as a solution

$$u(\zeta,\tau) = f[\tau - v(u)\zeta] \tag{10}$$

in terms of an arbitrary function f. In Eqs. (9) and (10) v is the amplitude-dependent function leading to shock formation, which is given by

$$v(u) = -\Omega - 3\varepsilon\Omega^2 + \frac{3A}{2}(1+6\varepsilon\Omega)u - [6\varepsilon A^2 + N^2(\beta_1 + 2\beta_2)]u^2 + \frac{\varepsilon}{2u}g', \tag{11}$$

where the prime denotes differentiation with respect to u.

The integration of Eq. (5) can proceed after σ_ζ is determined as a function of u, which is accomplished by integration of Eq. (7) with respect to τ and differentiation with respect to ζ resulting in the expression

$$\sigma_\zeta = k + A(\Omega - 3\varepsilon\Omega^2)u - \tfrac{3}{4}A^2(1+6\varepsilon\Omega)u^2 + \left(2\varepsilon A^3 + \frac{N^2A}{3}(\beta_1 + 2\beta_2)\right)u^3 - \varepsilon A u_{\tau\tau}, \tag{12}$$

where k is an arbitrary constant. The amplitude function u is next obtained by substitution of Eqs. (7) and (12) into Eq. (5) resulting in

$$u_{\tau\tau}(1-\alpha u) - \alpha u_\tau^2 + a_1 u + a_2 u^2 + a_3 u^4 = 0, \tag{13}$$

with the coefficients given by

$$\begin{aligned} \alpha &= \frac{6\varepsilon A}{1+6\varepsilon\Omega}, \\ a_1 &= -\frac{2}{1+6\varepsilon\Omega}\left(k + \kappa + \frac{\Omega^2}{2} + \varepsilon\Omega^3\right), \\ a_2 &= \frac{A^2}{2} - \frac{2N^2}{1+6\varepsilon\Omega}(1+\beta_1\Omega), \\ a_3 &= \frac{2}{1+6\varepsilon\Omega}\left[\frac{2AN^2}{3}(\beta_1 - \beta_2) - \varepsilon A^3\right]. \end{aligned} \tag{14}$$

With $1 - \alpha u$ as an integrating factor, the first integral of Eq. (13) is easily found to be

$$u_\tau^2(1-\alpha u)^2 + a_1u^2 - \frac{2\alpha a_3}{3}u^3 + \frac{a_2}{2}u^4 + \tfrac{2}{5}(a_3 - \alpha a_2)u^5 - \frac{\alpha a_3}{3}u^6 = 0, \tag{15}$$

and factoring the five polynomial terms as $(1-\alpha u)^2 \times u^2(r_1 + r_2u + r_3u^2)$, we obtain the first integral

$$u_\tau = u\sqrt{r_1 + r_2u + r_3u^2}. \tag{16}$$

Here the coefficients are given by $r_1 = a_1$, $r_2 = 4\alpha a_1/3$, and $r_3 = (a_2/2) + \frac{5}{3}\alpha^2 a_1$ with the factorization condition

$$2a_3 + 3\alpha a_2 + 10\alpha^3 a_1 = 0. \tag{17}$$

Finally the arbitrary function g is determined by use of this first integral together with Eq. (8) giving

$$g = r_1u^2 + 2r_2u^3 + 3r_3u^4. \tag{18}$$

Before looking at the nature of the different solutions we will use Eq. (18) in Eq. (11) to obtain an expression for the function v that involves only fiber parameters and the arbitrary constants k, A, and Ω:

$$v = a_1 - \Omega - 3\varepsilon\Omega^2 + \left[\frac{3A}{2}(1+6\varepsilon\Omega) + 4\alpha a_1\right]u - [6\varepsilon A^2 + N^2(\beta_1 + 2\beta_2) - 3a_2 - 10\alpha^2 a_1]u^2, \tag{19}$$

where use has been made of the factorization condition to eliminate a_3.

At this point Eq. (16) is integrated yielding the initial pulse structure ($\zeta = 0$), and the pulse shapes for arbitrary values of ζ are found from Eq. (10). In the following we consider only localized pulses resulting in the condition $a_1 < 0$ giving the following solution:

$$u(\zeta, \tau) = \frac{-4a_1 Z}{(Z + r_2)^2 + 4r_3}, \tag{20}$$

with $Z = \exp(\pm\eta\sqrt{-a_1})$ and the variable τ has been replaced by $\eta = \tau - v(u)\zeta$. Defining u_0 to be the maximum amplitude of the pulse, we can use Eq. (20) to find the following expression relating coefficients:

$$u_0^2 a_2 + \tfrac{10}{3}\alpha a_2 + 5\alpha^3 a_1 = 0, \tag{21}$$

which will be used later to obtain the values of A and a_1.

The evolution of two types of pulses is described by Eq. (20): The first that we will consider is the solitary wave that occurs when v is independent of u, and the second exhibits pulse distortion and shock formation occurring in the more general case when v depends on u. The linear and quadratic terms in Eq. (19) lead to pulse distortion, so the solitary wave solution requires additional conditions relating parameters resulting in these two terms equating to zero. Equating the linear term to zero gives a simple expression for a_1 and use of Eq. (21) gives the coefficient a_1 in terms of ε and Ω:

$$a_1 = -\frac{1}{16\varepsilon}(1 + 6\varepsilon A\Omega). \tag{22}$$

Next the quadratic term is equated to zero giving an expression for a_2:

$$a_2 = 2\varepsilon A^2 + \frac{N^2}{2}(\beta_1 + 2\beta_2) - \tfrac{10}{3}\alpha^2 a_1, \tag{23}$$

which can be used along with the factorization condition given by Eq. (17) to find relations between the parameters. For example, it is possible to calculate a simple expression for A,

$$A^2 = \frac{9\varepsilon N^2(\beta_1 + 2\beta_2) + 4}{6\varepsilon(1 - 9\varepsilon)}, \tag{24}$$

as well as a more complicated expression relating the parameters A, Ω, and κ by use of Eq. (23) along with the definition of a_2. The constant term in Eq. (19) is related to the pulse speed, and the form of the solution is given by Eq. (20). Notice that this solution is asymmetric owing to the nonzero value of the parameter A. The symmetric solution of Ref. [22] is obtained from Eq. (16) if $A = 0$ which results in $r_2 = 0$, and the sech function will be a particular solution.

For the less restrictive case when Eq. (19) has a u dependence, we being by using the factorization condition (17) and Eq. (21) to find values for the parameters A and a_1. This is done for the special case when $\Omega = 0$, $\kappa = 0$, $\beta_1 = \beta_2$, and $N^2 = 1$, resulting in the quadratic system

$$6^3\varepsilon^2 A^2 a_1 + \tfrac{1}{2}A^2 - \tfrac{36}{10} = 0,$$
$$a_1^2(1 + 8\varepsilon A u_0) + u_0^2\left(1 - \frac{A^2}{4}\right) - 60\varepsilon^2 A^2 u_0^2 a_1 = 0, \tag{25}$$

which can be solved for a_1 and A in terms of ε and u_0. These equations have been solved numerically to obtain solutions. However, only one solution is used in the following because the others result in either complex values for A, contrary to the initial requirement that u is a real function, or they result in positive values for a_1 contrary to the assumption that the solution is localized. The solution of Eq. (25) for the case when $\varepsilon = 0.1$ is shown in Fig. 1 illustrating a dependence of A and a_1 on u_0. It is also remarked that there is a maximum u_0 above which all roots of Eq. (24) are imaginary; when $\varepsilon = 0.1$ this value is 0.1203.

Next pulse distortion is seen by solution of Eq. (20) for u using v given by Eq. (19). This is done numerically for various values of ζ again using the special values of parameters $\Omega = 0$, $N^2 = 1$, $\beta_1 = \beta_2$, and $\varepsilon = 0.1$ with A given by the values in Fig. 1. The calculated pulse shapes for different values of ζ are shown in Fig. 2 where one can clearly notice the ζ-dependent asymmetry leading to shock formation occurring at the critical value of ζ found to be 15.04. From these data it is noticed that there is an intensity-dependent shock formation distance ζ_c defined to be the propagation distance where the maximum slope of the pulse goes to infinity, which is given by

$$\zeta_c = -\frac{1}{(f_\eta v_u)_{\max}}, \tag{26}$$

where the subscript refers to the maximum value. The shock formation distance was obtained from the above equation for the values of parameters $\Omega = 0$, $N^2 = 1$, and $\beta_1 = \beta_2$ again using the values of A from Fig. 1. The

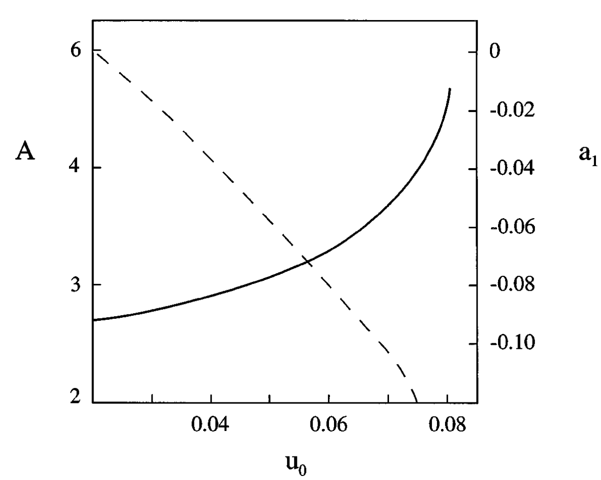

FIG. 1. The quantities A (solid curve) and a_1 (dashed curve) versus the initial pulse amplitude u_0.

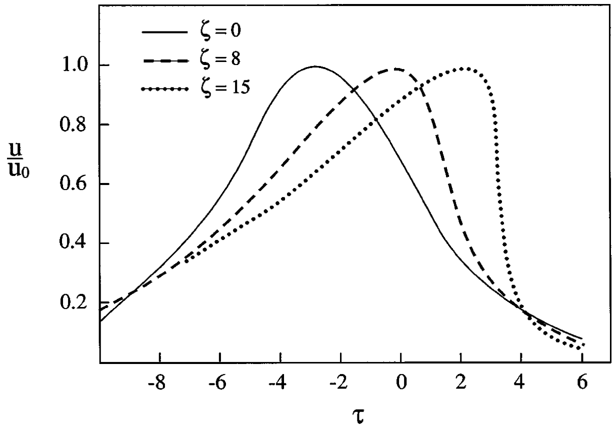

FIG. 2. Plot of the normalized pulse intensity versus τ showing the initial pulse, $\zeta = 0$ (solid curve), with self-steepening illustrated, $\zeta = 8$ (dashed curve), and $\zeta = 15$ (dotted curve).

quantity $(f_\eta v_u)_{\max}$ was found numerically with $\varepsilon = 0.1$ as a function of u_0 and these data are used to calculate the u_0 dependence of the shock formation length which is illustrated in Fig. 3. As in Ref. [16] it is noticed from Fig. 3 that the shock formation distance is an increasing function of the pulse intensity, but owing to the complicated form of this initial pulse shape in Eq. (20) it is not possible to derive a simple analytical expression for ζ_c.

In conclusion, solitary wave and shock solutions of the HONLS equation are obtained from the assumption that the phase function is a linear function of u. The solitary wave solutions are more general than previous solitary wave solutions because they are asymmetric as a result of the phase function. Furthermore, they exhibit a carrier wave phase shift across the pulse, which is not a property of the previous HONLS solitary wave solutions. Using Eq. (2) an estimate of the propagation distance required for shock formation in a typical fiber is found. For the values $D_2 = 20\ \mathrm{ps}^2/\mathrm{km}$, $T_0 = 1$ ps, and $u_0 = 0.1$ we estimate a value of z_c to be about 0.75 km.

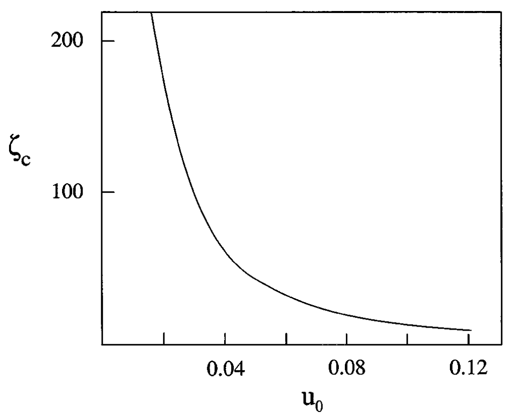

FIG. 3. Plot of the critical propagation distance ζ_c versus the initial pulse amplitude.

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