Statistics of Wave Dynamics in Random Media

A. Z. Genack,[1] P. Sebbah,[2] M. Stoytchev,[1] and B. A. van Tiggelen[3]
[1]Department of Physics, Queens College of the City University of New York, Flushing, New York 11367
[2]Laboratoire de Physique de la Matière Condensée/CNRS, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
[3]Laboratoire de Physique et Modélisation des Milieux Condensés, CNRS/Maison de Magistères, Université Joseph Fourier, B.P. 166, 38042 Grenoble Cedex 9, France
(Received 21 August 1998; revised manuscript received 23 October 1998)

We measure the joint distribution of the energy transmission coefficient ε_{ab} and the single channel delay time τ_{ab} for microwave radiation propagating through a random medium in the limit of zero pulse bandwidth. For fixed ε_{ab} the distribution of τ_{ab} is Gaussian with variance inversely proportional to ε_{ab}. In contradistinction to τ_{ab}, the dynamic matrix element $\varepsilon_{ab}\tau_{ab}$ has substantial long-range correlation with frequency shift. We present a new dynamic theory for a Gaussian process, which is in excellent agreement with experiment. [S0031-9007(98)08297-0]

PACS numbers: 42.25.Dd

The focus of statistical optics and mesoscopic physics has been on steady state transmittance quantities such as intensity, total transmission, and conductance. In practice, however, all experiments are bounded in time and it is natural to consider the statistics of dynamical aspects of propagation. Key variables are the "single channel delay time" τ_{ab} and the energy transmission coefficient ε_{ab} for a pulse $I_a^{\rm in}(t)$ incident in spatial mode a and scattered into outgoing mode b, defined in terms of the time-dependent transmission coefficient $I_{ab}(t)$ as $\tau_{ab} = \int dt\, I_{ab}(t)t/\int dt\, I_{ab}(t)$, and $\varepsilon_{ab} = \int dt\, I_{ab}(t)/\langle\int dt\, I_a^{\rm in}(t)\rangle$. The interplay between the single channel delay time and the energy transmission coefficient is best expressed in terms of the joint probability distribution $P(\varepsilon_{ab}, \tau_{ab})$ and the correlation functions with frequency of ε_{ab}, τ_{ab}, and $\varepsilon_{ab}\tau_{ab}$. In the limit of small bandwidth, the single channel delay time is found to approach the spectral derivative $d\phi_{ab}/d\omega \equiv \phi'_{ab}$ of the phase accumulated by the field as it propagates through the sample [1], while the energy transmission coefficient ε_{ab} approaches the single channel static transmission coefficient $I_{ab}(\omega)$. To initiate our study, we consider the important limiting case of narrow bandwidth, which is commonly taken in discussions of group velocity and phase delay in homogeneous media. The concept of delay times has been discussed previously by Eisenbud [2] and Wigner [3] as ways of quantifying the duration of collisions. Smith [4] has introduced the delay matrix to treat the many channel problem. Statistical aspects of dynamics have been considered for chaotic scattering [5–9], with potential applications to coherent electrons in mesoscopic capacitances [10] and to microwave cavities. Here we consider statistics of dynamics of microwave radiation in random media.

An interplay between the statistics of I and ϕ' can be expected by noting that the phase is indeterminate at nulls in the transmitted intensity speckle pattern. The phase jumps by π when moving through a null in the speckle pattern [11,12]. Thus the magnitude of ϕ' can be particularly large for low values of the intensity when nulls in the speckle pattern pass near a detector. The strong correlation between static intensity and dynamic delay time is important since the single channel delay time weighted by the intensity $W_{ab} = I_{ab}\phi'_{ab}$ [4] is the appropriate quantity for the delay time averaged over channels. For example, the average delay for photons in mode a is $W_a = (1/2N)\Sigma_b I_{ab}\phi'_{ab}$, and the sum over all pairs of modes is the Wigner delay time $W = (1/2N)\Sigma_{ab} I_{ab}\phi'_{ab}$, which is the density of states of the sample multiplied by $\pi/2N$ [13]. When the modes a and b are restricted, respectively, to modes on the incident and output sides of the sample, W_a and W can be considered as the dynamic equivalents, respectively, of the total transmission T_a and the dimensionless conductance T, whose ensemble average value is g. Enhanced fluctuations in T_a and T as well as enhanced spectral correlations arise as a result of extended spatial correlation [14–18]. Here we report the observation of a long-range C_2 correlation [14,18] in dynamics. This is seen in both the frequency correlation function of W_{ab}, as well as in its probability distribution. In contrast, the statistics of the single channel delay time ϕ'_{ab} is found to be remarkably well described by the Gaussian or C_1 approximation [19].

We report here measurements of the microwave field transmission coefficient $t_{ab}(\omega) = \sqrt{I_{ab}}\,\exp(i\phi_{ab})$ through a sample of randomly positioned $\frac{1}{2}$-inch polystyrene spheres between 18 and 19 GHz using a Hewlett-Packard HP8722C network analyzer. The sample has a volume filling fraction of 0.52 and is contained in a 5-cm-diam copper tube. New configurations are produced after each spectrum is taken by rotating the tube briefly about its axis. Ten thousand spectra are taken in a sample of length $L = 100$ cm with frequency steps of 100 kHz. Measurements at $L = 50$ cm and $L = 200$ cm are taken for 6000 sample configurations with frequency steps of 625 kHz. The transport mean free path is approximately 7 cm [20]. The transmitted field excited by an incident

0031-9007/99/82(4)/715(4)$15.00

pulse $E_a^{\rm in}(t)$ is obtained by Fourier transforming into the time domain the product of the frequency domain Fourier transforms of the incident pulse and the field transmission coefficient $E_a^{\rm in}(\nu)t_{ab}(\nu)$. The single channel delay time $\tau_{ab}(\nu, \Delta\nu)$ is computed for each configuration for an incident pulse with carrier frequency near $\nu = \omega/2\pi = 18.1$ GHz and for various bandwidths $\Delta\nu$. From this we compute the probability distribution for τ_{ab} normalized to its ensemble average value $\hat{\tau}_{ab} = \tau_{ab}/\langle\tau_{ab}\rangle$ for a number of bandwidths. We find that the distribution broadens with decreasing bandwidth and approaches a limiting distribution as $\Delta\nu \to 0$. We have previously shown that the single channel delay time in the limit of long pulses is equal to ϕ'_{ab} [1]. The limiting distribution thus corresponds to the distribution of ϕ'_{ab}.

The probability distribution of the normalized single channel delay time $\hat{\phi}' \equiv \phi'_{ab}/\langle\phi'_{ab}\rangle$ is shown in Fig. 1. Figure 2 shows that the distributions of $\hat{\phi}'$ for *fixed* I_{ab} are well fit by Gaussian functions. Their variances are found to be inversely proportional to $\hat{I}$ as shown in the inset of the figure. The spectral correlation functions of single channel delay time $\hat{\phi}'$ and the weighted single channel delay time W_{ab} are shown in Fig. 3. The correlation function of $\hat{\phi}'$ falls exponentially with frequency, whereas the correlation function of W_{ab} falls as $1/\sqrt{\Delta\nu}$. These decays are reminiscent, respectively, of the decays of the C_1 and C_2 terms in the intensity correlation function with frequency [14,15,18,19].

To our knowledge, neither "C_1" nor "C_2" theory has been developed previously for dynamic variables. We begin our calculations by considering the C_1 or Gaussian approximation for K complex field amplitudes $E_i = t(i)E_i^{\rm in}$ [21]. Their joint distribution is given by

$$P(E_1, \ldots, E_K) = \frac{1}{\pi^K \det \mathbf{C}} \exp\left(- \sum_{i,j=1}^{K} \bar{E}_i C_{ij}^{-1} E_j \right), \qquad (1)$$

where $C_{ij} = \langle E_i \bar{E}_j \rangle$ is the normalized Hermitian variance matrix, $\langle E_j \bar{E}_j \rangle = 1$. In our case, the index i labels K different frequencies for a given channel transition ab. For small frequency difference $\omega_1 - \omega_2 = \omega$, we can make the expansion $C_{12} = 1 + ia\omega + b\omega^2 + \mathcal{O}(\omega^3)$, where a and b can be calculated from diffusion theory [15,18] (involving diffusion constant D, sample length L, and the absorption length L_a), or can simply be measured.

Probability distributions can be derived for $K = 2$ using a change of variables $E_j = A_j \exp(i\phi_j)$. As $\omega \to 0$, the stochastic variables become $I = A^2$, $\phi' \equiv d\phi_{ab}/d\omega$, $R \equiv d \log A_{ab}/d\omega$, and ϕ_{ab}. Integrating out ϕ_{ab} and R yields for the joint distribution

$$P(I, \phi') = \sqrt{\frac{I}{\pi Q a^2}} \exp(-I) \exp\left[-\frac{I}{Q a^2} (\phi' - a)^2 \right]. \qquad (2)$$

The distribution depends on the dimensionless parameter $Q \equiv -2b/a^2 - 1 > 0$. For constant intensity I, ϕ' is normally distributed with standard deviation $\Delta\phi'/\langle\phi'\rangle = \sqrt{Q/2I}$. The distributions for ϕ'_{ab} and W_{ab} follow from Eq. (2):

$$P\left(\hat{\phi}' \equiv \frac{\phi'}{\langle\phi'\rangle} \right) = \frac{1}{2} \frac{Q}{[Q + (\hat{\phi}' - 1)^2]^{3/2}}; \qquad (3)$$

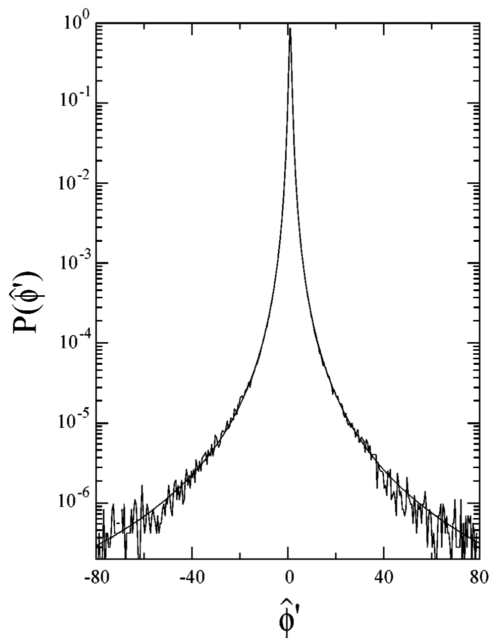

FIG. 1. Probability distribution function of the normalized single channel delay time $\hat{\phi}' \equiv \phi'_{ab}/\langle\phi'_{ab}\rangle$, in transmission. The solid line is the theoretical prediction using the measured value $Q = 0.31$.

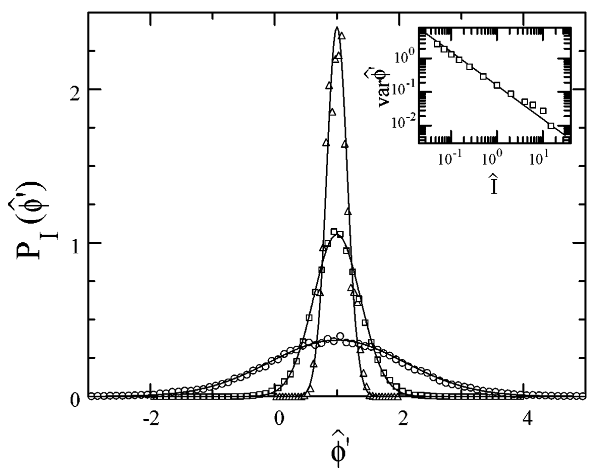

FIG. 2. Probability distributions for the normalized local delay time $\hat{\phi}'$ in transmission of a tube with length $L = 100$ cm for fixed values of the normalized intensity $\hat{I}$; circles: $\hat{I} = 0.1$; squares: $\hat{I} = 1.0$; triangles: $\hat{I} = 10$. The Gaussian curves denote the theoretical predictions for $Q = 0.31$. Inset: Normalized variance $(\Delta\hat{\phi}')^2$ of the single channel delay time at fixed normalized intensity $\hat{I}$. The solid line gives the theoretical prediction for $Q = 0.31$.

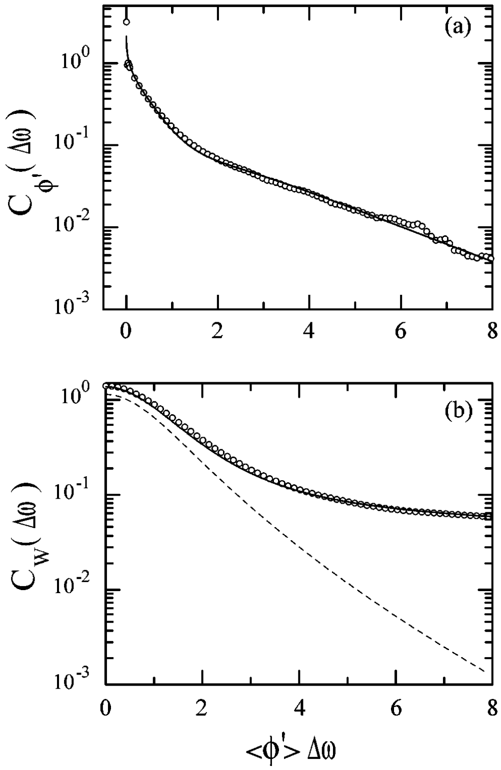

FIG. 3. Normalized frequency correlation functions for (a) single channel phase delay ϕ'_{ab} and (b) single channel phase delay weighted by intensity W_{ab}. Solid lines denote theoretical predictions using the value $L/L_a = 2.5$ obtained from the measurement of the field correlation function. The ϕ'_{ab}-correlation function exhibits no significant long-range correlation, and is well described by the Gaussian approximation. The W_{ab}-correlation function exhibits a long-range $1/\Delta\omega^{1/2}$ tail. The solid line is the theoretical prediction for $C_1 + C_2$ using $g = 6.0$; the dashed line shows C_1 only.

$$P\left(\hat{W} \equiv \frac{W}{\langle W\rangle}\right) = \frac{1}{\sqrt{Q+1}} \exp\left(\frac{-2|\hat{W}|}{\theta(\hat{W}) + \sqrt{Q+1}}\right). \tag{4}$$

The distribution $P(\hat{\phi}')$ decays algebraically in a way similar to delay times in single channel chaotic cavities [9].

Correlation functions at two close frequencies can be obtained from Eq. (1) with $K = 4$ at the frequencies $\nu \pm \omega/2 \pm \Omega/2$ in the limit $\omega \to 0$. The calculation of the normalized frequency correlation function of $W_{ab}(\nu)$ involves integrations that can all be done analytically,

$$\langle \hat{W}_{ab}(\nu - \Omega/2)\hat{W}_{ab}(\nu + \Omega/2)\rangle_c = \frac{1}{2a^2}\left[|C'(\Omega)|^2 - \mathrm{Re}\, C(\Omega)\bar{C}''(\Omega)\right] \equiv F_1(\Omega)\,. \tag{5}$$

The details of the frequency correlation function of $\phi'_{ab}(\nu)$ must be worked out numerically and are left to future publications. Figure 4 shows Q, $F_1(\Omega = 0)$, and the average phase delay calculated from the diffusion approximation [22].

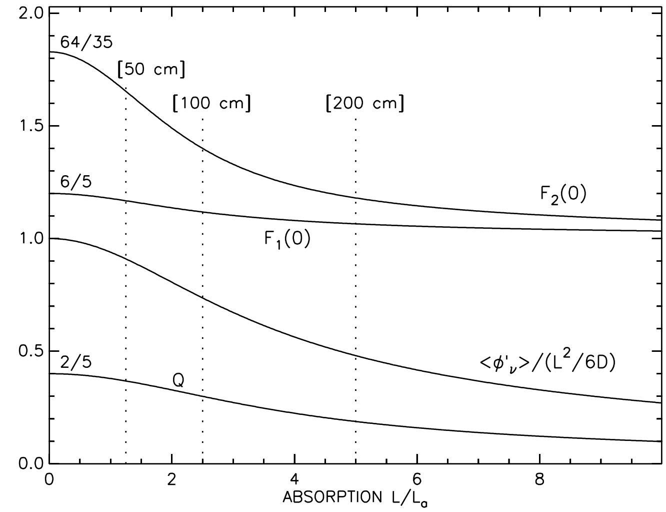

FIG. 4. Several parameters which appear in the statistics of the delay time are shown as functions of the absorption L/L_a in transmission through a thick slab of length L, calculated from the diffusion approximation. The function $F_1(0)$ denotes the short range "C_1" contribution to $\langle W^2\rangle - \langle W\rangle^2$; $\frac{1}{g}F_2(0)$ is the non-Gaussian "C_2" contribution. Q is the dimensionless parameter determining the probability distribution of the quantities W and ϕ'; the average delay time $\langle\phi'_\nu\rangle$ has been normalized to the diffuse traversal time $L^2/6D$. The dashed lines indicate the positions of the samples used in the measurements: $Q = 0.37$, $Q = 0.31$, and $Q = 0.18$ have been explicitly measured.

In order to compare the measured distribution $P(\hat{\phi}')$ to theory, we must find the single parameter Q. A measurement of the field frequency correlation function with frequency steps that are 1% of the correlation frequency for the $L = 100$ cm sample gives $Q = 0.31 \pm 0.01$. The solid line in Fig. 1 is a plot of Eq. (3) using this value for Q and is in agreement with experiment over 7 decades in $\hat{\phi}'_{ab}$.

The theoretical dependence of the variance of $\hat{\phi}'_{ab}$ upon $\hat{I}$ is shown as the straight line in the inset of Fig. 2, and again found to be in excellent agreement with experiment. The agreement of the measurements of $P(\hat{\phi}')$ and the conditional probability $P(\hat{\phi}'/\hat{I})$ with C_1 theory shows that long-range correlation does not influence the statistics of $\hat{\phi}'$. This is confirmed by the agreement shown in Fig. 3a of the spectral correlation function of $\hat{\phi}'$ with the C_1 prediction obtained numerically using Eq. (1) and the measured value $L/L_a = 2.5$. In the limit $\Delta\omega \to 0$ this correlation function is predicted to diverge logarithmically. We see in Fig. 3a that this correlation function rises to a value of 3.5, which is limited by the finite frequency step and the measurement noise.

The relationship of both the dimensionless conductance g and the Wigner delay time W to the density of states [23,24] leads us to anticipate that long-range correlations will be displayed in the single channel dynamic quantities W_{ab}, as it is in the static intensity I_{ab} [14–18]. The presence of long-range correlation for a single polarization component was seen in the increasing deviation from a

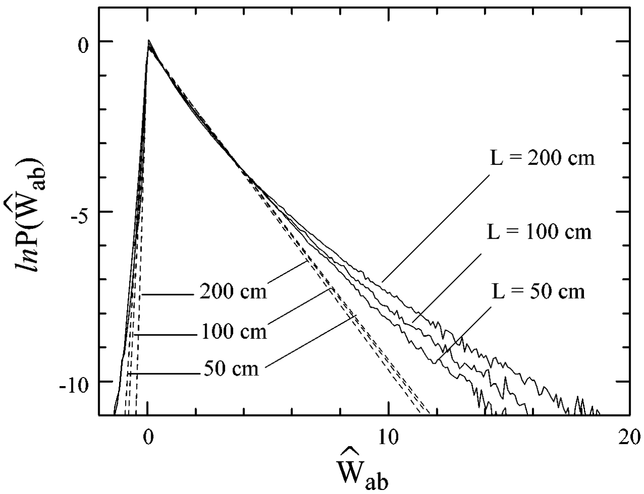

FIG. 5. Probability distributions of the normalized single channel phase delay weighted by intensity $\widehat{W}_{ab} = I_{ab}\phi'_{ab}/\langle I_{ab}\phi'_{ab}\rangle$, for three sample lengths: $L = 50$ cm, $L = 100$ cm, and $L = 200$ cm. The dashed lines denote the C_1 theory using $Q = 0.37$, $Q = 0.31$, and $Q = 0.18$.

negative exponential statistics in the tail of $P(\hat{I}_{ab})$ [16,25–29], as well as in a significant C_2 contribution to the intensity correlation function with frequency, decaying as $1/\sqrt{\Delta\nu}$ [14,17,18]. In Fig. 5 we compare the measured distribution of $\widehat{W}_{ab}$ for various lengths with the two sided exponential given by Eq. (4). The increasing deviations found in the tails of the distributions as L increases—with corresponding decrease of the dimensionless conductance g—signals the presence of long-range correlation. This is indicated as well in the spectral correlation function of $\widehat{W}_{ab}$, shown in Fig. 3b. The dashed lines show the prediction of Eq. (4). The difference between experiment and C_1 theory is of the order $g^{-1}/\sqrt{\Delta\nu}$ in the tail of the correlation function. To understand this tail, the C_2 contribution $F_2(\Omega)$ of the spectral correlation function of $\widehat{W}_{ab}$ has been calculated from the theory of Berkovits and Feng [22], using the Hikami box of Nieuwenhuizen and van Rossum [30]. The sum $F_1(\Omega) + (1/g)F_2(\Omega)$ is shown as the solid line in Fig. 3b, where $g \approx 6$ is known from measurements of the static intensity distribution in this sample [26]. The results are in excellent agreement with theory without the use of any free parameters.

In conclusion, we have reported the measurement and calculation of key distributions and correlation functions of single channel dynamics in the limit of narrow-band incident pulses. This study forms the foundation for the statistics of arbitrary pulses. These results reveal the interplay between the delay time and intensity, which is associated with the structure of the transmitted speckle pattern. They also show that the weighted single channel delay time is not a self-averaging quantity and that large mesoscopic fluctuations in delay times can be expected.

We thank Olivier Legrand, Frédéric Faure, Roger Maynard, and Gabriel Cwilich for discussions. This work was supported by the National Science Foundation under Grants No. DMR 9632789 and No. INT9512975 and the Groupement de Recherches POAN.

[1] P. Sebbah, O. Legrand, B. A. van Tiggelen, and A. Z. Genack, Phys. Rev. E **56**, 3619 (1997); P. Sebbah, O. Legrand, and A. Z. Genack, Phys. Rev. E (to be published).

[2] L. Eisenbud, Ph.D. thesis, Princeton University, 1948.

[3] E. Wigner, Phys. Rev. **98**, 145 (1955).

[4] F. T. Smith, Phys. Rev. **118**, 349 (1960); **119**, 2098 (1960).

[5] J. G. Muga and D. M. Wardlaw, Phys. Rev. E **51**, 5377 (1995).

[6] P. Šeba, K. Zyczkowski, and J. Zakrewski, Phys. Rev. E **54**, 2438 (1996).

[7] P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Phys. Rev. Lett. **78**, 4737 (1997).

[8] Y. V. Fyodorov and H. J. Sommers, Phys. Rev. Lett. **76**, 4709 (1996).

[9] V. A. Gopar, P. A. Mello, and M. Büttiker, Phys. Rev. Lett. **77**, 3005 (1996).

[10] M. Büttiker, H. Thomas, and A. Prêtre, Phys. Lett. A **180**, 364 (1993).

[11] J. F. Nye and M. V. Berry, Proc. R. Soc. London Ser. A **336**, 165 (1974); M. V. Berry, J. Phys. A **11**, 27 (1978).

[12] I. Freund, N. Shvartsman, and V. Freilikher, Opt. Commun. **101**, 247 (1993); N. Shvartsman and I. Freund, Phys. Rev. Lett. **72**, 1008 (1994).

[13] G. Iannacone, Phys. Rev. B **51**, 4727 (1995).

[14] M. Stephen and G. Cwilich, Phys. Rev. Lett. **59**, 285 (1987).

[15] S. Feng, C. Kane, P. Lee, and A. D. Stone, Phys. Rev. Lett. **61**, 834 (1988).

[16] N. Garcia and A. Z. Genack, Phys. Rev. Lett. **63**, 1678 (1989).

[17] J. F. de Boer, M. P. van Albada, and A. Lagendijk, Phys. Rev. Lett. **64**, 2787 (1990).

[18] A. Z. Genack, N. Garcia, and W. Polkosnik, Phys. Rev. Lett. **65**, 2129 (1990).

[19] B. Shapiro, Phys. Rev. Lett. **57**, 2168 (1986).

[20] A. Z. Genack, J. H. Li, N. Garcia, and A. A. Lisyansky, in *Photonic Band Gaps and Localization,* edited by C. M. Soukoulis (Plenum, New York, 1993).

[21] J. W. Goodman, *Statistical Optics* (John Wiley, New York, 1985).

[22] R. Berkovits and S. Feng, Phys. Rep. **238**, 135 (1994). For the tube geometry in transmission we use $C(\Omega) = z/\sinh z$ with $z = \sqrt{i\Omega L^2/D + (L/L_a)^2}$.

[23] B. L. Alt'shuler and B. I. Shklovskii, Sov. Phys. JETP **64**, 127 (1986).

[24] E. R. Muccioli, R. A. Jalabert, and J. L. Pichard, J. Phys. I (France) **10**, 1267 (1997).

[25] A. Z. Genack and N. Garcia, Europhys. Lett. **21**, 753 (1993).

[26] M. Stoytchev and A. Z. Genack, e-print Cond-mat/9805233.

[27] E. Kogan, M. Kaveh, R. Baumgartner, and R. Berkovits, Phys. Rev. B **48**, 9404 (1993).

[28] Th. M. Nieuwenhuizen and M. C. W. van Rossum, Phys. Rev. Lett. **74**, 2674 (1995).

[29] E. Kogan and M. Kaveh, Phys. Rev. B **52**, R3813 (1995).

[30] Th. M. Nieuwenhuizen and M. C. W. van Rossum, Phys. Lett. A **177**, 102 (1993).