Deduction of the Quantum Numbers of Low-Lying States of 6-Nucleon Systems Based on Symmetry

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The inherent nodal structure in the wave functions of 6-nucleon systems is investigated using group theory. The existence of a group of six low-lying states composed of mainly an $L = 0$ component is deduced. In addition to the $\{4, 2\}$ spatial permutation symmetry, the $\{2, 2, 2\}$ symmetry is found to be also important for the low-lying states. [S0031-9007(98)08131-9]

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In the context of nuclear physics, when the number of nucleons N is small, say $N \leq 4$, the nuclear structure can be determined by exact diagonalization. On the other hand, there exist a few approximate methods based on model theories for systems with large N. The essential difficulty arises when N is neither small nor large. To be specific, for the intermediate values of $N \sim 5-10$, the convergence of shell model calculations is usually poor (i.e., the calculated energies do not converge rapidly when the number of basis functions is increased [1]), while in the cluster model it is necessarily complicated to include many different cluster configurations (a partition of the N nucleons is associated with a cluster configuration; eight configurations are used in a ^{6}Li calculation [2]). Although many efforts have been made to investigate the $N \sim 5-10$ systems [1–7], no general theory has been established because of the complexity due to so many degrees of freedom.

The purpose of this paper is to explore a new approach using group theory to study the effects of the inherent symmetries. The nodal structure of the few-body wave functions will be investigated in detail in order to obtain certain important features of the wave functions and the energy spectra before actually solving the Schrödinger or Faddeev equations. In particular, the quantum numbers of the low-lying states of 6-nucleon systems will be determined in this paper.

It has been shown in Ref. [8] that (i) the ground state of ^{4}He is composed of components mainly with zero orbital angular momentum ($L = 0$), while all resonances below the $2n + 2p$ threshold have mainly $L = 1$. Since the excitation energies of all resonances are large ($\geq$20 MeV), the increase of L leads to a great increase in energy. Thus the components with a larger L are not important for the low-lying states. (ii) All of the above states are composed mainly of nodeless components (i.e., their internal wave functions do not contain nodal surfaces; therefore, internal oscillations have not yet been excited). Since the 6-nucleon and 4-nucleon systems have comparable size and weight, it is reasonable to assume that the $L = 0$ nodeless components should again dominate the low-lying spectra of 6-nucleon systems.

It has also been found that a specific class of nodal surface may be imposed on the wave functions by symmetry [9]. We denote Ψ as an eigenstate and A as a geometric configuration. In some cases, A may be invariant to a set of operators O_i ($i = 1, 2, \ldots$); i.e., $O_i A = A$. For example, when A is a regular octahedron (OCTA) of a 6-body system (in Fig. 1a), it is invariant to a rotation about a fourfold axis of the OCTA by 90° together with a cyclic permutation of the four particles. In this case we have

$$\hat{O}_i \Psi(A) \equiv \Psi(O_i A) = \Psi(A) . \tag{1}$$

From the inherent transformation property of Ψ (i.e., the property with respect to rotation, space inversion, and permutation), Eq. (1) can always be written in matrix form (as we shall see) and appears to be a set of homogeneous linear equations. It is well known that homogeneous equations do not always have nonzero solutions. Thus Eq. (1) imposes a very strong constraint on Ψ so that Ψ may be zero at A. This indicates that there exists a class of nodal surfaces which are determined by the intrinsic symmetry of the system instead of by the dynamics. This specific class of nodal surfaces is denoted as the inherent nodal surfaces (INS).

The INS are closely related to the geometric symmetry. For a 6-nucleon system, the OCTA has the strongest geometric symmetry. We denote, in Fig. 1a, o as the center of mass (c.m.) of the six particles, o' as the c.m. of the particles 2, 3, and 5, and i'-j'-k' as the body frame. Referring to $R_\delta^{\vec{v}}$ as a rotation about the axis along the vector $\vec{v}$ by an angle δ (in degrees), p_{ij} as an interchange of the particles i and j, and P as a space inversion, one can easily see that the OCTA is invariant to operations

$$O_1 = p(1432) R_{-90}^{k'} , \tag{2}$$

where $p(1432)$ denotes a cyclic permutation,

$$O_2 = p_{13} p_{24} p_{56} P , \tag{3}$$

$$O_3 = p(253) p(146) R_{-120}^{oo'} , \tag{4}$$

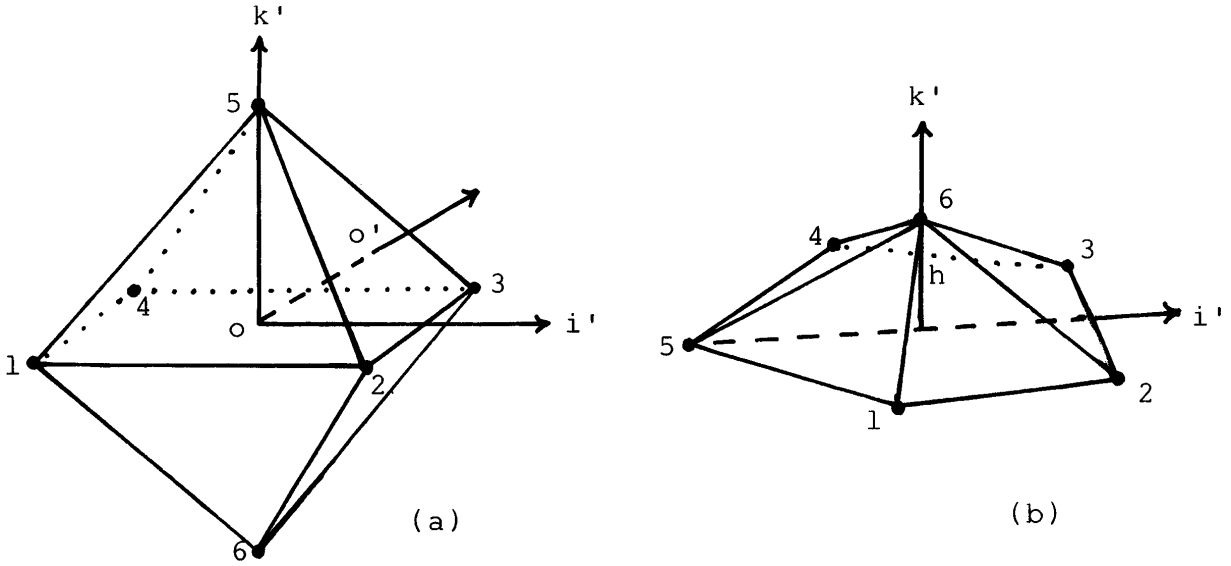

FIG. 1. The two favorite geometric shapes of a 6-nucleon system. (a) The regular octahedron, (b) the regular pentagon pyramid.

where oo' is the vector connecting o and o', and

$$O_4 = p_{14}p_{23}p_{56}R^{i'}_{180}. \tag{5}$$

It is certain that the OCTA is also invariant to some other operations, e.g., the $p(152)p(364)R^{oo''}_{-120}$ (where o'' is the c.m. of the particles 1, 2, and 5). However, since $R^{oo''}_{-120} = R^{k'}_{-90}R^{oo'}_{-120}R^{k'}_{90}$, no additional constraints are thereby introduced. Thus, one can prove that the operations O_1 to O_4 are sufficient to specify the constraints arising from symmetry.

Let an eigenstate of a 6-nucleon system with a given total angular momentum J, parity Π, and total isospin T, be written as

$$\Psi = \sum_{L,S,\lambda} \Psi_{LS\lambda}, \tag{6}$$

$$\Psi_{LS\lambda} = \sum_{iM} C^{JM_J}_{LM,SM_S} F^{\lambda i}_{LSM} \chi^{\tilde{\lambda} i}_{SM_S}, \tag{7}$$

where S is the total spin and $M(M_S)$ is the Z component of the $L(S)$. The L and S are coupled to J. $F^{\lambda i}_{LSM}$ is a function of the spatial coordinates. It is the ith basis function of the representation λ of the permutation group S_6. The $\chi^{\tilde{\lambda} i}_{SM_S}$ is a basis function in the spin-isospin space with given S, T and belonging to the representation $\tilde{\lambda}$, the conjugate of λ. The λ representations allowed in (6) are listed in Table I, depending on the S and T [10].

TABLE I. The allowed representation λ in Eq. (6).

S	T	λ
0	0	$\{1^6\},\{2,2,1,1\},\{3,3\},\{4,1,1\}$
1	0	$\{2,1^4\},\{3,1^3\},\{2,2,2\},\{3,2,1\},\{4,2\}$
2	0	$\{2,2,1,1\},\{3,2,1\}$
3	0	$\{2,2,2\}$
0	1	$\{2,1^4\},\{3,1^3\},\{2,2,2\},\{3,2,1\},\{4,2\}$
1	1	$\{1^6\},\{2,1^4\},2\{2,2,1,1\},\{3,1^3\},$ $2\{3,2,1\},\{3,3\},\{4,1,1\}$
2	1	$\{2,1^4\},\{2,2,1,1\},\{3,1^3\},\{2,2,2\},\{3,2,1\}$
3	1	$\{2,2,1,1\}$

The $F^{\lambda i}_{LSM}$ can be expanded in the body frame i'-j'-k' as

$$F^{\lambda i}_{LSM}(123456) = \sum_{Q} D^{L}_{QM}(-\gamma,-\beta,-\alpha) \times F^{\lambda i}_{LSQ}(1'2'3'4'5'6'), \tag{8}$$

where α,β,γ are the Euler angles to specify the rotation. D^L_{QM} is the well-known Wigner function. Q is the projection of the L along the k' axis. The (123456) and $(1'2'3'4'5'6')$ specify that the coordinates are relative to the fixed frame and body frame, respectively.

Since the $F^{\lambda i}_{LSQ}$ spans a representation of the rotation group, space inversion group, and permutation group, the invariance of the OCTA to the operators O_1 to O_4 leads to four sets of equations. For example, from

$$\hat{O}_1 F^{\lambda i}_{LSQ}(A) \equiv F^{\lambda i}_{LSQ}(O_1 A) = F^{\lambda i}_{LSQ}(A) \tag{9}$$

[(A) denotes that the coordinates are given at an OCTA] for a Q with $Q \le L$, we get

$$\sum_{i'} [g^{\lambda}_{ii'}(p(1234))e^{-i(\pi/2)Q} - \delta_{ii'}]F^{\lambda i'}_{LSQ}(A) = 0, \tag{10}$$

where $g^{\lambda}_{ii'}$ are the matrix elements of the representation λ, which are known from group theory (see, for example, Ref. [11]). From O_2, O_3, and O_4, we have

$$\sum_{i'} [g^{\lambda}_{ii'}(p_{13}p_{24}p_{56})\Pi - \delta_{ii'}]F^{\lambda i'}_{LSQ}(A) = 0, \tag{11}$$

$$\sum_{Q'i'} \Big\{ g^{\lambda}_{ii'}[p(235)p(164)] \sum_{Q''} D^{L}_{QQ''}(0,\theta,0)e^{-i(2\pi/3)Q''} \times D^{L}_{Q'Q''}(0,\theta,0) - \delta_{ii'}\delta_{QQ'} \Big\} F^{\lambda i'}_{LSQ'}(A) = 0, \tag{12}$$

$$\sum_{Q'i'} [(-1)^L g^{\lambda}_{ii'}(p_{14}p_{23}p_{56})\delta_{\bar{Q}Q'} - \delta_{ii'}\delta_{QQ'}]F^{\lambda i'}_{LSQ'}(A) = 0. \tag{13}$$

where $\bar{Q} = -Q, \theta = \arccos\sqrt{1/3}$. Equations (10)–(13) are the equations that the $F^{\lambda i}_{LSQ}(A)$ have to fulfill. In some cases there is at least one common nonzero solution(s) (i.e., not all of the $F^{\lambda i}_{LSQ}(A)$ are zero) for all of these equations. However, in some other cases, there are no common nonzero solutions. In the latter case, the $\Psi_{LS\lambda}$ has to be zero at the OCTA configurations despite their size and orientation. Accordingly, an INS emerges and the OCTA is inaccessible. It is evident that the above equations depend only on the L, Π, and λ and therefore the existence of the INS is independent of the dynamics.

The results of the $L = 0$ components are given directly (in Table II), neglecting the evaluation of nonzero solutions of linear equations.

While the wave functions are strongly constrained at the OCTA, they are less constrained in the neighborhood of the OCTA. For example, when the shape in Fig. 1a is prolonged along k', it is called a prolonged octahedron. This shape (denoted by B) is invariant to O_1, O_2, and O_4, but not to O_3. Hence, the $F^{\lambda i'}_{LSQ'}(B)$ should fulfill only the Eqs. (10), (11), and (13), but not (12). Evidently, a common nonzero solution of Eqs. (10)–(13) is necessarily a common solution of Eqs. (10), (11), and (13). Thus, if a $\Psi_{LS\lambda}$ is nonzero at an OCTA, it remains nonzero in the neighborhood. In other words, an OCTA-accessible component is inherently nodeless in the domain surrounding the OCTA.

Another shape with a stronger geometric symmetry of a 6-nucleon system is the regular centered-pentagon [(C-PENTA), plotted in Fig. 1b with the $h = 0$]. The C-PENTA is invariant to (i) a rotation about k' by 72° ($\frac{2\pi}{5}$) together with a cyclic permutation of the five particles of the pentagon, (ii) a rotation about k' by 180° together with a space inversion, and (iii) a rotation about i' by 180° together with $p_{14}p_{23}$. These invariances lead to constraints embodied by three sets of homogeneous equations, and therefore the accessibility of the C-PENTA can be identified as listed in Table II. As before, a C-PENTA-accessible component is inherently nodeless in the domain surrounding the C-PENTA, in particular, at the pentagon pyramid (Fig. 1b).

In addition to the OCTA and the C-PENTA, there are other regular shapes, e.g., hexagons, where the INS might emerge. However, among the 15 pairwise bonds linking the nucleons, 12 bonds can be optimized in an OCTA, 10 in a pentagon pyramid, but only 6 in a hexagon. Therefore, in the neighborhood of the hexagon (and other regular shapes), the total potential energy is considerably higher. Since the wave functions of low-lying states are distributed mainly in the domains with a relatively lower potential energy, we shall concentrate only on the domains surrounding the OCTA and the C-PENTA.

Referring to Table II, one can find that, when a $\Psi_{LS\lambda}$ has $(L^{\Pi}\lambda) = (0^+\{6\})$, $(0^+\{4,2\})$, or $(0^+\{2,2,2\})$, it can access both the OCTA and the C-PENTA. These and only these $\Psi_{LS\lambda}$ are inherently nodeless in the two most important domains. They should then be the dominant components of the low-lying states. All other states must contain at least one INS which may result in a large increase in energy. From Table I, it is clear that the $(0^+\{6\})$ component is completely forbidden, whereas the $(0^+\{4,2\})$ component is allowed to the $[S,T] = [1,0]$ and $[0,1]$ states, and the $(0^+\{2,2,2\})$ component is allowed to the $[S,T] = [1,0]$, $[3,0]$, $[0,1]$, and $[2,1]$ states.

Let us study the contribution of these two favorite $L = 0$ components. When $[S,T] = [1,0]$, both the $\{4,2\}$ and $\{2,2,2\}$ components are available. Therefore, two $J^\pi = 1^+$ partner states would be generated (as shown in Table III). Each of them is mainly a specific mixture of the $\{4,2\}$ and $\{2,2,2\}$ components. Similarly, when $[S,T] = [0,1]$, two partner states with $J^\pi = 0^+$ would also be generated. Since only the $\{2,2,2\}$ is available for $[S,T] = [3,0]$ and $[2,1]$, there would be one $J^\pi = 3^+$ state ($T = 0$) and one $J^\pi = 2^+$ state ($T = 1$). Therefore, a total of six low-lying states dominated by the $L = 0$ inherently nodeless components are predicted [cf. Table III, where the L, S, and λ are only the quantum numbers of the dominant component(s)].

It is expected that these low-lying states should be split by the nuclear force. Because of the interference of the $\{4,2\}$ and $\{2,2,2\}$ components, there would be a gap between the partner states. Based on experimental data, Ajzenberg-Selove analyzed the low-lying spectrum of ^{6}Li [12]. The result is replotted in Fig. 2 and listed

TABLE II. The accessibility of the OCTA and the C-PENTA to the $L^\pi = 0^+$ and 0^- wave functions with different spatial permutation symmetry λ. The figures in the blocks are the numbers of independent nonzero solutions.

λ	0^+ OCTA	0^+ C-PENTA	0^- OCTA	0^1 C-PENTA
$\{6\}$	1	1	0	0
$\{5,1\}$	0	1	0	0
$\{4,2\}$	1	1	0	0
$\{3,3\}$	0	1	0	0
$\{2,2,2\}$	1	1	1	0
$\{2,2,1,1\}$	0	1	0	0
$\{2,1^4\}$	0	1	0	0
$\{1^6\}$	0	1	0	0
$\{3,2,1\}$	0	2	0	0
$\{4,1,1\}$	0	0	0	0
$\{3,1^3\}$	0	0	1	0

TABLE III. Prediction of the quantum numbers of low-lying states of the 6-nucleon systems based on a symmetry analysis. The last column shows the energies (in MeV) of the states of ^{6}Li taken from Ref. [12].

S	T	J	π	L	λ	E
1	0	1	+	0	$\{4,2\}$ and $\{2,2,2\}$	0
1	0	1	+	0	$\{4,2\}$ and $\{2,2,2\}$	5.65
3	0	3	+	0	$\{2,2,2\}$	2.19
0	1	0	+	0	$\{4,2\}$ and $\{2,2,2\}$	3.56
0	1	0	+	0	$\{4,2\}$ and $\{2,2,2\}$	
2	1	2	+	0	$\{2,2,2\}$	5.37

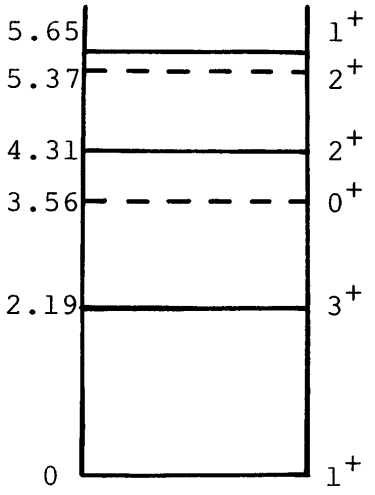

FIG. 2. The low-lying energy levels (in MeV) of ^{6}Li nucleus from Ref. [12]. The $T = 0$ levels are shown by the solid lines, and $T = 1$ by the dashed lines. The J^Π values are given on the right-hand side.

in Table III. Although our analysis is based simply on symmetry, the results of the two analyses are close to each other.

For the $T = 0$ states, the two expected partners with $J^\pi = 1^+$ ($[S, T] = [1, 0]$) are found in Fig. 2 with an energy split as expected. The split is so large (5.65 MeV) that the lower state becomes the ground state, while the higher one becomes the highest state of this group. The expected $J^\pi = 3^+$ state ($T = 0$) has been found in Fig. 2 at 2.19 MeV. Nevertheless, the 4.31 MeV state with $J^\pi = 2^+$ and $T = 0$ in Fig. 2 does not appear in our analysis. Perhaps this state is dominated by PENTA-accessible but OCTA-inaccessible components $(0^+\{3, 2, 1\})$ and $(0^+\{2, 2, 1, 1\})$ with $S = 2$, or dominated by the $L = 1$ components, or other origins to be clarified.

For the $T = 1$ states, one of the expected partners with $J^\pi = 0^+$ ($[S, T] = [0, 1]$) is found in Fig. 2 at 3.56 MeV. However, the other state (maybe considerably higher) has not yet been identified in Ref. [12]. Nonetheless, if this state exists, its spatial structure would be similar to that of the $T = 0$ state at 5.65 MeV. The third expected $T = 1$ state has been found in Fig. 2 at 5.37 MeV with exactly the predicted quantum number $J^\pi = 2^+$.

Based on symmetry analysis, we have determined the quantum numbers of the low-lying states of 6-nucleon systems. In addition to the $\{4, 2\}$, the $\{2, 2, 2\}$ is also found to be important. Our explanation is very different from that based on the shell model. For example, according to our analysis, the $J^\pi = 3^+$ state at 2.19 MeV has mainly the quantum numbers $S = 3$, $L = 0$, and $\lambda = \{2, 2, 2\}$. However, it is recognized in the shell model calculation that this state has mainly $S = 1$, $L = 2$, and $\lambda = \{4, 2\}$ [4]. Nevertheless, the following points should be noticeable: (i) This 3^+ state is above the $(\alpha + d)$ channel, but the width is very narrow (only 24 keV). Since the open channel has $S = 1$ and $\lambda = \{4\} \otimes \{2\}$ ($\otimes$ denotes the outer product), if this state has $S = 1$ and $\lambda = \{4, 2\}$, it would couple strongly with the open channel. Alternatively, if $\lambda = \{2, 2, 2\}$, the channel is closed. The experimentally observed very narrow width supports the latter. (ii) In Ref. [4], two types of calculations, namely the variational Monte Carlo (VMC) and the Green's function Monte Carlo (GFMC), were performed. Based on the shell model, the trial wave function for the $J^\pi = 3^+$ state is initiated with the $\{4, 2\}$ component in both calculations. It is noted that if there is a shortcoming in the trial wave function, it can be cured in the GFMC, but cannot in the VMC. It turns out that the calculated energy of the 3^+ state in VMC is much higher than the experimental value. This discrepancy does not arise from the interactions (because the result of the GFMC is good), but from the trial wave function. Thus the effectiveness of the shell model for very light nuclei is questionable.

Up to now, the most accurate wave functions for 6-nucleon systems were those obtained in Ref. [4] by the GFMC. It was found in this calculation that the kinetic energies of the low-lying states of ^{6}Li decrease when excitation energies increase. This is due to a slight increase in size during excitations. On the other hand, this is strong evidence that the internal oscillations have not yet been excited. In other words, the main components of these states do not contain nodal surfaces. This finding supports the basic assumption of this paper.

In summary, we have carried out a model-independent analysis for 6-nucleon systems based on basic symmetry. It is shown that a class of wave functions exists which is inherently nodeless. These wave functions are the most important building blocks which constitute the low-lying states. The identification of these favorite components plays a key role in understanding the low-lying spectra. Our method provides a valuable alternative to the study of light nuclei. The general idea is presumably applicable to other few-body systems as well, and thus enriches our understanding of these important systems.

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