Anderson Localization due to a Random Magnetic Field in Two Dimensions

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Results of large-scale numerical simulations are reported on the Anderson localization in a two-dimensional square lattice tight-binding model with random flux. Localization lengths, fluctuations of the conductance, and the density of states are computed for quasi-one-dimensional geometry. Numerical results indicate that the model exhibits the same critical behavior as the one studied by Gade [Nucl. Phys. **B398**, 499 (1993)]. It is argued that all the states except a zero-energy state are localized and the density of states has a singularity in the center of the band. The energy scale below which the density of states increases is found to be extremely small ($\lesssim 10^{-2}$). [S0031-9007(98)08263-5]

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It is general wisdom that noninteracting electrons are localized in two-dimensional (2D) disordered systems [1]. There are, however, some well-known exceptions to this rule. These include electrons having strong spin-orbit coupling [2] and integer quantum Hall systems [3]. Recent studies have shown that 2D Dirac fermions with random gauge field offer another exception to the rule [4,5]. For a model of 2D nonrelativistic fermions subjected to random magnetic field with zero mean, the existence of delocalized states has been a subject of debate.

The random flux model, in which static magnetic field is randomly distributed with zero mean, got much attention recently in connection with the gauge field theory of high-T_c superconductivity [6] and the composite-fermion theory of the half-filled Landau level [7]. It has been controversial, however, whether this model has a delocalized state [8]. On the one hand, several numerical and analytical studies concluded that all the states are localized and belong to the unitary class of the scaling theory [9–14]. On the other hand, a different conclusion that there are delocalized states near the center of the band was reached by other people [15–22]. One source of the discrepancy in numerical works is the extremely large localization length near the band center, making it difficult to decide whether or not states are localized from numerical data of finite-size systems.

In this paper I present various numerical results obtained through the largest numerical simulations performed so far for the square lattice tight-binding model subjected to random flux with zero mean. The results indicate that a state at the band center ($E = 0$) is not localized. This is reminiscent of the integer quantum Hall system. There is, however, a crucial difference: the density of states (DOS) is found to be divergent at $E = 0$ in the random flux case. This behavior is similar to the 1D and 2D random hopping models [23,24], and a crucial role is played by a special particle-hole symmetry relating a state of energy E with a state of energy $-E$. The random flux model is argued to be in the same universality class as a model studied by Gade [25]. Although this was already anticipated in [4,20], this Letter reports for the first time that the random flux model shares a hallmark of the Gade model, i.e., the divergence of the DOS at $E = 0$.

The Hamiltonian of the tight-binding model is

$$H = -\sum_j \sum_{k=1}^{M} (c_{j+1,k}^\dagger c_{j,k} + e^{i\theta_{j,k}} c_{j,k+1}^\dagger c_{j,k} + \text{H.c.}), \tag{1}$$

where $c_{j,k}$ is the annihilation operator of a fermion on site (j,k). The random magnetic flux is introduced through the random Peierls phase $\theta_{j,k}$ in the hopping matrix element. The magnetic flux $\phi_{j,k} = \theta_{j,k} - \theta_{j-1,k}$ takes a random number in $-\pi p \le \phi_{j,k} \le \pi p$ with a uniform distribution. The parameter p is set to be 1, except in Fig. 4 (shown below). Numerical calculations are done for samples that have quasi-1D geometry of width M in the y direction and of length L in the x direction ($M \ll L$). A periodic boundary condition is imposed in the y direction ($c_{j,M+1} = c_{j,1}$), whereas open boundary conditions are assumed in the x direction for most of the calculation. For even M the lattice can be divided into A and B sublattices. For each eigenfunction ψ_E with energy E, changing the sign of ψ_E on every site of, say, the A sublattice yields a new eigenfunction ψ_{-E} with energy $-E$ [20,26]. This symmetry relating the $\pm E$ states holds for each disorder configuration. For odd M, however, the particle-hole symmetry is absent under the periodic boundary condition.

The localization length is calculated from the exponential decay of the retarded Green's function obtained by using the standard recursive algorithm [27]: $\langle \ln \|G_E^r(1,k;L,k')\| \rangle \sim -L/\lambda_M$, where $\|G\|$ and $\langle\,\rangle$ denote the norm of G and the ensemble average, respectively. Figure 1 shows the quasi-1D localization length λ_M normalized by M as functions of M and E. The typical length of quasi-1D samples used in the calculation is 3×10^5, 4×10^5, and 8×10^5 for $M = 32$, 64, and 128, respectively. Furthermore, ensemble average is taken, typically, over 70 (20) samples for $M \le 64$ ($M = 128$) to reduce

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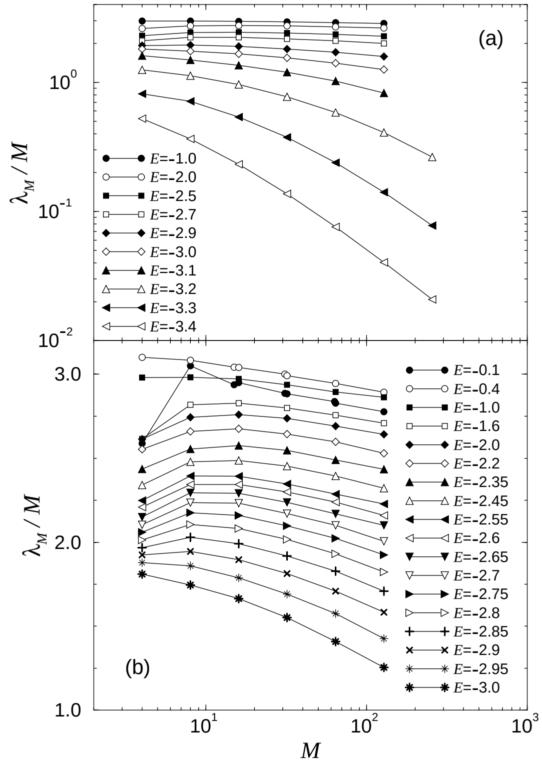

FIG. 1. Localization length for quasi-1D geometry calculated for $M = 4$, 8, 16, 32, 64, 128, and 256. Additional data of $M = 15$ and 31 are shown for $E = -0.1$ and -0.4. In (b) the statistical error for the data at $M = 128$ is about the same size as the symbols, whereas for smaller M the error bar is much smaller than the symbols. The kink at small M in the $E = -0.1$ data should be a finite-size effect.

the statistical error. The quality of the numerical data is therefore greatly improved from the earlier numerical results [9,13,16,18,20]. Clearly the states near the band edges ($|E| > 3.0$) are localized [Fig. 1(a)]. Figure 1(b) shows λ_M/M decreases as M increases, suggesting that the states with $|E| \geq 0.1$ are all localized in the 2D limit.

The localization lengths of the quasi-1D wires are expected to satisfy the one-parameter scaling $\lambda_M/M = f(\xi/M)$, where ξ is the localization length in 2D. The scaling indeed holds as shown in Fig. 2 [28]. The scaling curve quantitatively agrees with the earlier results of Refs. [9] and [10]. The agreement with the latter work is somewhat surprising in that a network model is used in [10] which is an effective model in the semiclassical limit. The 2D localization length ξ grows exponentially and reaches 10^6 lattice spacings at $E = -2.55$; see inset.

Figure 3 shows λ_M/M versus M at $E = 0$ [29]. There is a striking even-odd effect in the $E = 0$ data, as noticed earlier in Refs. [20,30]. A new finding here is that $\lambda_M/M|_{E=0}$ stays almost constant for odd M while it gradually increases for even M, suggesting that $\lambda_M/M \rightarrow \text{const}(>0)$ as $M \rightarrow \infty$. This may imply that $\psi_{E=0}$ is a critical or multifractal wave function, as suggested by Miller and Wang [20]. By contrast, at $|E| = 0.1$, there is little even-odd oscillation [Fig. 1(b)], and λ_M/M is a decreasing function of M. The importance of the particle-hole symmetry can also be seen by examining the effects of on-site disorder, which breaks the symmetry. The on-site disorder is introduced by adding a term $\sum_{j,k} \epsilon_{j,k} c^\dagger_{j,k} c_{j,k}$ to H, where $\epsilon_{j,k}$ are taken to be randomly distributed in the interval $[-w/2, w/2]$. Figure 3 clearly shows that, in the presence of the on-site disorder, λ_M's of even and odd M's merge together at $M \approx M_c$ and decrease for $M > M_c$. The crossover width is $M_c \approx 64$ for $w = 0.4$ and increases for smaller w. These results demonstrate the crucial role of the symmetry on the $E = 0$ wave function.

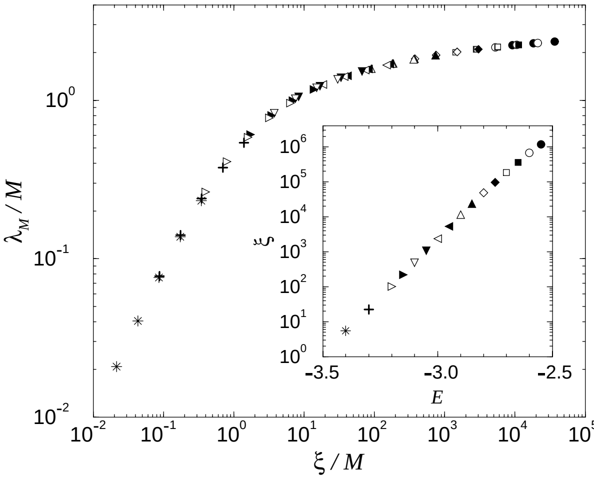

FIG. 2. Scaling curve obtained from the data for $-3.4 \leq E \leq -2.55$. For $-3.0 < E \leq -2.55$ only the data of $M \geq 32$ are used. Inset: Localization length versus energy.

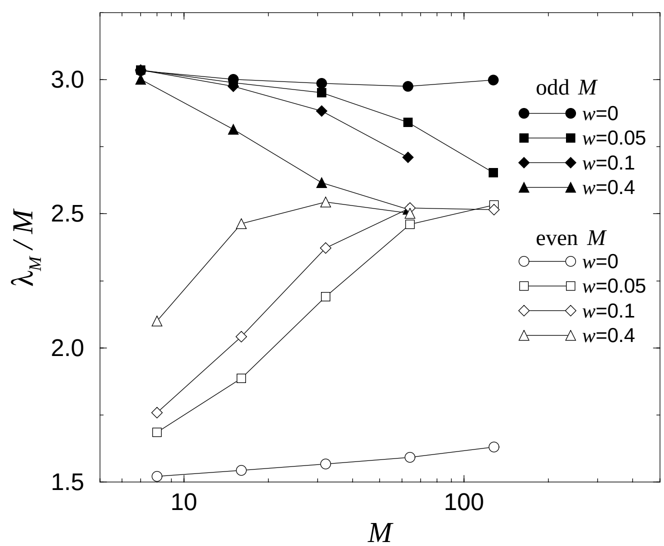

FIG. 3. Localization length at $E = 0$ as functions of M and the on-site disorder w. The filled symbols represent data for $M = 7$, 15, 31, 63, and 127. The open symbols are data for $M = 8$, 16, 32, 64, and 128.

The states away from the band center belong to the unitary class. This can be verified by calculating fluctuations of two-terminal conductance as a function of L. For this purpose, perfect leads are attached to both ends of quasi-1D wires, and the transmission matrix t is calculated from the Green's function G_E^r. The dimensionless conductance g is then obtained from the Landauer formula, $g = \mathrm{Tr}(tt^\dagger)$. Figure 4 shows $\mathrm{var}\, g = \langle g^2 \rangle - \langle g \rangle^2$ for $M = 32$, averaged over 2×10^4 samples. For $|E| = 0.1$, $\mathrm{var}\, g$ is calculated for $p = 1$ and 0.2 without the on-site disorder. Almost identical $\mathrm{var}\, g$ versus L/λ_M curves are obtained for $|E| = 1.0$ and 0.02 as well. A thin line in Fig. 4 shows $\mathrm{var}\, g$ of the unitary ensemble calculated in the limit $M \ll L$ by Mirlin *et al.* [31] using the supersymmetric σ model approach. Notice that, except for the peaks at $L < 0.5\lambda_M$, the numerical results of $|E| = 0.1$ are indistinguishable from the thin line (unitary ensemble). The discrepancy occurs only for $L \lesssim M$, where the samples are no longer quasi-one dimensional. The numerical curve of $p = 0.2$ is closer to the analytic result because $M/\lambda_M|_{p=0.2} \ll M/\lambda_M|_{p=1}$. These results clearly show that for $|E| \geq 0.1$ and $p \geq 0.2$ the wave functions belong to the unitary class.

The variance of g has a different L/λ_M dependence at $E = 0$ for even M; see the inset of Fig. 4. Without the on-site disorder, for each L/λ_M, $\mathrm{var}\, g$ of $E = 0$ is larger than $\mathrm{var}\, g$ of $E \neq 0$ [32]. This clearly shows that, when $w = 0$, the zero-energy state does not belong to the unitary class. The on-site disorder, however, drives $\psi_{E=0}$ back to the unitary class, as shown by the long-dashed line ($p = 1$ and $w = 0.2$) in the inset of Fig. 4. These observations are consistent with Fig. 3.

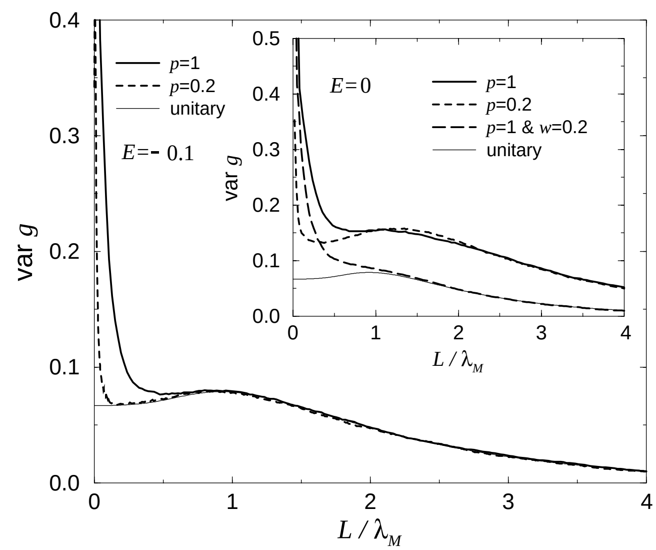

FIG. 4. Variance of g as a function of the length of the disordered region at $E = -0.1$. $\mathrm{var}\, g$ approaches 0 for L shorter than mean free path, although invisible in this scale. Inset: Variance of g at $E = 0$. The thin curves are the analytic result for the unitary ensemble [31], where $\mathrm{var}\, g \to 1/15$ as $L \to 0$ (with $M \ll L$). For both figures $M = 32$.

It seems quite natural to assume that the states belonging to the unitary class in the quasi-1D geometry remain to be in the same class as $M \to \infty$. This would mean that all the states away from the band center are localized. A state at $E = 0$, if it exists, should not be localized in 2D. It follows both from the recent result [29] that the state at $E = 0$ is delocalized for odd M under open boundary conditions in the y direction and from the numerical data in Fig. 3. The delocalization of the zero-energy state, which also implies the divergence of ξ towards the band center, is inferred by requiring that the 2D limit ($M \to \infty$) should be independent of the boundary conditions [33] and of the parity of M. The delocalization at the band center is a consequence of the particle-hole symmetry as in the random hopping model [23] and will be ruined by the on-site disorder.

As pointed out in [4,20], the random-flux model has the same symmetry property as the Gade model, and it is natural to expect that the two models share the same critical behavior. In the Gade model the localization length diverges towards the band center, where the DOS $\rho(E)$ is also divergent as $\rho(E) \sim \exp(-c\sqrt{\ln|1/E|})/|E|$ (c: constant) [25]. The characteristic energy scale below which the singularity of the DOS manifests itself is then $E_c = \exp(-c^2)$, which can be extremely small depending on c. This may explain why no singularity was found in $\rho(E)$ before [9,20,26,34]. To find the presumably weak singularity, I computed the DOS with high accuracy using the recursive method [35]. In this calculation a small imaginary number was added to the energy ($E \to E + i\gamma$), instead of attaching perfect wires. This amounts to averaging $\rho(E)$ over the energy interval of order γ. Figure 5 shows the DOS of a system of $L = 128\,000$ and $M = 64$ with $\gamma = 10^{-2}$. The overall shape of the DOS is similar to the one obtained by the retraced-path

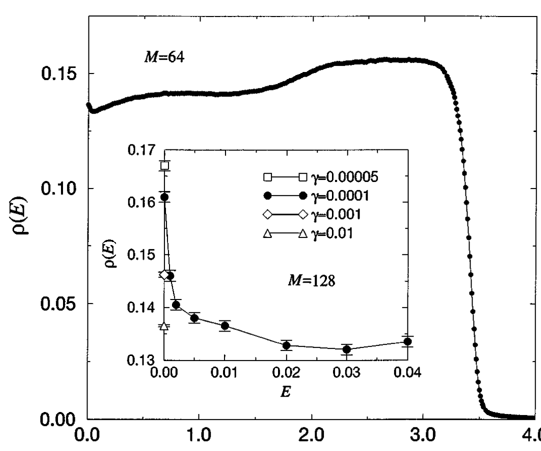

FIG. 5. Density of states of a system of $M = 64$ and $L = 128\,000$ calculated with $\gamma = 10^{-2}$. Inset: $\rho(E)$ of a system of $M = 128$ and $L = 64\,000$, $64\,000$, $204\,800$, and $256\,000$ for $\gamma = 10^{-2}$, 10^{-3}, 10^{-4}, and 5×10^{-5}, respectively. The nonvanishing DOS at $|E| \gtrsim 3.5$ may be an artifact of the smearing.

approximation [36]. Notice, however, the tiny peak centered at $E = 0$. Its height grows with smaller γ and larger M (inset), which is a clear signature of the divergent DOS. To determine the precise form of the singularity requires further investigation. It is important to note here that γ is kept large enough to smear out the microscopic structure in $\rho(E)$ near $E = 0$. Because of the level repulsion and of the particle-hole symmetry, $\rho(E)$ vanishes at $E = 0$ for even M [37]. It is expected that, in the limit $M \to \infty$, the dip in the DOS at $E = 0$ disappears and $\rho(E)$ diverges at $E = 0^+$, in analogy with the 1D random-hopping model with an even number of sites [23]. The moderate smearing due to γ helps reveal the diverging behavior.

The discovery of the divergent DOS at $E = 0$ establishes the connection between the lattice random flux model and the Gade model. The critical behavior of the latter model is closely related to the model of Dirac fermions with random gauge field [4,5], which has the same particle-hole symmetry. This supports the conclusion based on the symmetry argument that a state at the band center is the only delocalized state for any p ($0 < p \leq 1$) in the absence of the on-site disorder. In models without the particle-hole symmetry, all the states should be localized, in agreement with [10–12,14].

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