Localization of Spin Waves in the Quantum Hall Ferromagnet

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The topological nature of the quantized Hall plateau has a number of remarkable consequences. For the quantum Hall ferromagnet (QHF), it leads to the identification of the topological charge density of a spin distortion with the associated electrical charge density. Spin waves may couple to a scalar disorder potential via their topological density. This interaction is very similar to minimal coupling of quantum particles to a random flux. It leads to the localization of spin waves in the QHF. We derive a low-energy description of the system in terms of a nonlinear sigma-model of unitary supermatrices. A possible experimental signature of these effects in photoemission is suggested. [S0031-9007(99)09462-4]

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The ground state of a two-dimensional electron gas (2DEG) at exact filling of a single Landau level (filling fraction $\nu = 1$) is strongly ferromagnetic. The properties of this quantum Hall ferromagnet (QHF) are profoundly affected by the topological nature of the quantized Hall plateau. In a quantum Hall state, there is a commensuration between the magnetic flux through the 2DEG and the electrical charge density. This leads to the identification of the topological charge density of a spin distortion with an associated electrical charge density [1]. This identity may be understood as follows: the Berry phase induced by adiabatic transport through a spin distortion may be reproduced by the Aharonov-Bohm phase induced by a fictitious magnetic flux, proportional to the topological density of the spin distortion. In this way, commensuration between flux and charge in the QH state implies commensuration of electrical and topological charge densities. This is the distinguishing feature of the QHF. As a consequence, the elementary excitations formed as the filling fraction is moved slightly away from $\nu = 1$ are electrically and topologically charged objects known as Skyrmions [1,2]. In addition, spin waves may couple to a scalar disorder potential via their topological density [3]. This disorder interaction is very similar to minimal coupling of quantum particles to a random flux. The spin wave system provides a novel realization of this intently studied problem [4]. Quantum particles localize in a random flux and one may expect the disorder potential to have a similar effect upon spin waves. In this Letter, we investigate this localization of spin waves by weak disorder. We use supersymmetry [5] to construct a low-energy description of the spin wave system in terms of a nonlinear sigma model of unitary supermatrices. All states of this model are localized in two dimensions [6]. This demonstrates explicitly the localization of spin waves in the QHF. A possible experimental signature of these effects in photoluminescence is suggested.

Our starting point is the continuum field theory of the QHF, proposed by Sondhi *et al.* [1];

$$S = \int d^2x\, dt\Big(\bar{\rho}\mathbf{A}[\mathbf{n}] \cdot \partial_t \mathbf{n} + \frac{\rho_s}{2}(\nabla \mathbf{n})^2 + \bar{\rho} g B n_z\Big) + \int d^2x\, dt\, U(\mathbf{x}) J_o(\mathbf{x}) + \int dt\, V[J_0(\mathbf{x})], \qquad (1)$$

$$J_\mu = \frac{e\nu}{8\pi}\epsilon_{\mu\nu\lambda}\mathbf{n} \cdot \partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n}, \qquad (2)$$

where $\mathbf{n}(\mathbf{x})$ is an O(3)-vector order parameter of unit length, describing the local polarization of the quantum Hall system. The first line of Eq. (1) is the usual continuum field theory of a ferromagnet. It consists of the Berry phase, spin stiffness, and Zeeman energy, respectively. $\mathbf{A}[\mathbf{n}]$ is the vector potential of a unit monopole in spin space, $\bar{\rho}$ is the electron density ($\bar{\rho} = 1/2\pi l_B^2$, where l_B is the magnetic length), ρ_s is the spin stiffness, and g is the Zeeman coupling, into which we have absorbed the electron spin and the Bohr magneton for ease of notation. The second line contains terms arising due to the identity of electrical and topological charge densities in the QHF. $U(\mathbf{x})$ is a scalar disorder potential and $V[J_0]$ is the Coulomb self-energy of a charge distribution $J_0(\mathbf{x})$. All of the new physics is due to these terms.

In the absence of disorder, the ground state field configuration of Eq. (1) is ferromagnetic. The long wavelength spin waves are those of the familiar Heisenberg ferromagnet. One may imagine that weak disorder will induce small fluctuations in the ground state charge distribution [and, therefore, in the spin polarization via Eq. (2)]. Such fluctuations would lead to a spatial variation of the spin-stiffness. This picture is incorrect. The QH state is incompressible and there is a gap to all static distortions in the charge density. The gap is the quasiparticle/Skyrmion energy. A scalar disorder potential induces no distortion in the charge density or spin distribution if its fluctuations are bounded by the Skyrmion energy. This incompressibility may be shown directly from Eq. (1) using a Bogomolny bound argument [7]. This property of the QHF is rather fortunate. It allows one to perform a spin wave expansion about the ferromagnetic ground state and to use perturbation theory in the disorder potential.

We wish to expand Eq. (1) in small fluctuations, $\mathbf{l} = (l_1, l_2, 0)$, about the ferromagnetic ground state, $\bar{\mathbf{n}} = (0, 0, 1)$; $\mathbf{n} = (l_1, l_2, \sqrt{1 - |\mathbf{l}|^2})$. The effective action, expanded to quadratic order in these fluctuations, is given by

$$S = \frac{1}{2} \int d^2x\, dt\, \bar{l} \left(\frac{\bar{\rho}}{2}\, \partial_t - \rho_s \nabla^2 - \bar{\rho} g B \right) l - i \int d^2x\, dt\, \frac{e\nu}{8\pi}\, U(\mathbf{x}) \epsilon_{ij} \partial_i \bar{l} \partial_j l\,. \quad (3)$$

We use the complex notation, $l = l_1 + il_2$, $\bar{l} = l_1 - il_2$. The first line of Eq. (3) is the usual Schrödinger effective action for spin waves in a continuum ferromagnet. The second line is due to the identity of charge and topological charge embodied in Eq. (2). It describes the interaction of spin waves with a scalar disorder potential. Spin wave interactions, both via the Coulomb potential and due to higher orders in the spin wave expansion, have been ignored. The grounds upon which one may do this are discussed in the penultimate paragraph. Throughout the bulk of the calculations presented in this paper, we will ignore the Zeeman term in Eq. (3). It is a simple matter to reintroduce it at the end of our calculations. To see the relationship of Eq. (3) to the effective action for Schrödinger particles in a random flux [4], one should integrate the interaction term by parts. The result looks like minimal coupling to a vector potential, $A_i = -(e\nu/16\pi)\epsilon_{ij}\partial_j U(\mathbf{x})$, aside from the absence of an $|\mathbf{A}|^2$ term. In the remainder of this Letter, we use the supersymmetry technique of Efetov [5] in order to consider the possibility of spin wave localization in the QHF.

The correlations in the disorder potential felt by the two-dimensional electron gas in GaAs heterostructures are conveniently modeled as follows [8]:

$$\langle\langle U_{\mathbf{q}} U_{\mathbf{q}'} \rangle\rangle = (2\pi)^2 \delta(\mathbf{q} + \mathbf{q}') \gamma' \frac{e^{-2|\mathbf{q}|d}}{|\mathbf{q}|^2}\,. \quad (4)$$

d is the width of the insulating spacer layer separating the electrons from the ionized donor impurities. γ' is a measure of the disorder strength and is related to the area density of donor impurities, n_d; $\gamma' = (e\sqrt{n_d}/2\epsilon)^2(\nu/8\pi)^2$. Notice that the disorder has long range correlations. To simplify our explicit calculation, we assume a Gaussian, δ-function correlated distribution for the disorder potential: $\langle\langle U_{\mathbf{q}} U_{\mathbf{q}'} \rangle\rangle = (2\pi)^2 \gamma \delta(\mathbf{q} + \mathbf{q}')$. In this case, the disorder strength is independent of the wave vector. We return to the more realistic correlations of Eq. (4) later.

The self-energy, calculated in the self-consistent Born approximation, is

$$\Sigma^R(\mathbf{p}, \omega) = \epsilon_{ij} \epsilon_{kl}$$

. (5)

In this diagram, full lines represent spin wave propagators, crossbars indicate spatial derivatives of these propagators and the dotted lines represent disorder correlations. The thick line represents a full spin wave propagator including the self-energy; $G^{-1} = G_0^{-1} + \Sigma$. The real part of the self-energy may be absorbed into a renormalization of the spin wave stiffness. The imaginary part is given by

$$\mathrm{Im}\Sigma^R(\mathbf{p}, \omega) = \gamma\, \mathrm{Im} \int \frac{d^2q}{(2\pi)^2} (\mathbf{p} \times \mathbf{q})^2 G^R(\mathbf{q}, \omega) = \frac{\gamma}{8\rho_s^2} \left(\frac{\bar{\rho}\omega}{2} \right) |\mathbf{p}|^2. \quad (6)$$

The second line of the above gives the solution to lowest order in the disorder strength. The disorder averaged spin wave Green's function is then

$$\langle G^R(\mathbf{p}, \omega) \rangle = \left(\rho_s |\mathbf{p}|^2 - \frac{\bar{\rho}\omega}{2} + i\, \frac{\bar{\rho}}{2\tau_\omega} \right)^{-1}, \quad (7)$$

where $\tau_\omega = 4\bar{\rho}\rho_s^3(\bar{\rho}\omega/2)^{-2}/\gamma$. In writing down this expression, we have replaced $|\mathbf{p}|^2$ in Eq. (6) by its on-shell value, $\bar{\rho}\omega/2\rho_s$. This is justified by Taylor expansion in powers of $\gamma\bar{\rho}\omega/16\rho_s^3$. $\gamma/\rho_s^2 \ll 1$ for weak disorder and $\bar{\rho}\omega/16\rho_s \ll 1$ at frequencies much less than the Skyrmion energy.

The most important point to notice here is that the scattering time diverges as the frequency goes to zero. The charge density of a superposition of spin waves is proportional to the second power of the momentum. Therefore, despite the disorder strength being the same for all momenta it has less effect upon low momenta. The reduction in charge density at low momenta also leads to a divergent lifetime in the case of the realistic disorder correlations of Eq. (4). This divergence is less rapid; $\tau'_\omega = 8\rho_s^2/\gamma'\omega$. The divergence of the scattering time has some important calculational consequences. The spectral weight is concentrated in a small energy range $1/2\tau_\omega \ll \bar{\rho}\omega/2$ of the bare pole at $\bar{\rho}\omega/2 = \rho_s|\mathbf{q}|^2$ as required for self-consistency of the perturbative expansion. This should be compared to the case of electrons scattering from a random scalar potential [5], where the bare pole of the Green's function is near to the Fermi energy, E_F, and the scattering rate, $1/\tau$, is constant. The validity of the perturbative expansion for electrons depends upon the smallness of the parameter $1/\tau E_F$. In the present case, the extra derivatives in the interaction vertex lead to the existence of a small expansion parameter without the existence of a Fermi surface.

Supersymmetry.—We now develop a low-energy theory for the interaction of spin waves with a weak disorder potential, using supersymmetric techniques. The main subtleties of the current problem are in the handling of the geometrical factors in the interaction. We refer the reader to the literature [5] for details of the supersymmetry itself. As noted previously, this problem is very similar to that of noninteracting particles in a random flux. An alternative derivation of the results presented here may be made using the techniques of Ref. [4].

We wish to determine the disorder averages of dynamical quantities involving $\langle G^A G^R \rangle$ and, therefore, introduce a four-component superfield, $\psi = (l^A l^R \chi^A \chi^R)$, in the usual way [5]. The superscripts, A/R, label advanced and retarded sectors, the fields χ are anticommuting. The presence of the commuting and anticommuting fields in ψ removes the need to write the partition function explicitly in the denominator of correlation functions and allows the disorder average to be performed immediately. The resulting Lagrangian for the superfield, ψ, is

$$\mathcal{L} = \int \left[-i\bar{\psi}\left(-\rho_s \nabla^2 - \frac{\bar{\rho}}{2}\tilde{\epsilon}\right)\psi + \gamma(\epsilon_{ij}\partial_i\bar{\psi}\partial_j\psi)^2 \right] d^2 r\,,$$

where $\tilde{\epsilon} = (\Omega + \omega/2)\mathbf{1} + (\omega/2 + i\delta)\Lambda$ and Λ is the diagonal supermatrix with elements $+/-$ in the advanced/retarded sectors. As is usual in the derivation of the supersymmetric sigma model, we assume that $\omega \ll \Omega$. Disorder averaging may induce cross correlations between the Green's functions only if the difference in frequencies is less than the scattering rate $\bar{\rho}\omega/2 \leq 1/2\tau_\Omega, 1/2\tau_{\Omega+\omega} \ll \bar{\rho}\Omega/2$, therefore this approximation is justified. Next, we decouple the quartic interaction, introduced by the disorder average, with a supermatrix field, Q_{ij}, where $i, j \in \{x, y\}$ and each element of the 2×2 matrix Q_{ij} is a 4×4 supermatrix:

$$\mathcal{L}_{\mathrm{int}}[\psi, Q_{ij}] = \int \frac{d^2 r}{2\tau_\Omega} \mathrm{Str}\left[\left(\frac{\bar{\rho}\Omega}{2\rho_s}\right)^{-1} \epsilon_{ki}\partial_k\bar{\psi}Q_{ij}\partial_j\psi + \frac{1}{32\rho_s} Q_{ij}^2\right].$$

The supertrace, Str, is as defined in Ref. [5] and summation over repeated spatial indices is implied. Integrating out the superfield, ψ, we obtain the following free energy functional for Q_{ij}:

$$F[Q_{ij}] = \int \mathrm{Str}\left[-\frac{1}{2}(\ln G^{-1}) + \frac{\bar{\rho}}{64\tau_\Omega \rho_s} Q_{ij}^2\right] d^2 r\,, \tag{8}$$

where $G(\mathbf{r}, \mathbf{r}', Q)$ is the supermatrix Green's function of the field ψ and satisfies the equation

$$\left(-\rho_s\nabla^2 - \frac{\bar{\rho}}{2}\tilde{\epsilon} + i\frac{\rho_s}{\tau_\Omega \Omega}\epsilon_{ki}[Q_{ij}\partial_j\partial_k + \partial_k Q_{ij}\partial_j]\right) \times G(\mathbf{r}, \mathbf{r}', Q) = i\delta(\mathbf{r} - \mathbf{r}')\,.$$

The saddle point equation for Q_{ij} is

$$Q_{ij} = \frac{16\rho_s^2}{\bar{\rho}\Omega}\int \frac{d^2 p}{(2\pi)^2}\epsilon_{ki}p_k p_j G(\mathbf{p}, Q_{ij})\,.$$

This is precisely the self-consistent Born equation that was solved previously to find ImΣ. The solution is

$$Q_{ij} = \epsilon_{ij} V \Lambda \bar{V}\,,$$

where V is an arbitrary, unitary supermatrix such that $V\bar{V} = 1$. The diagonal terms of Q_{ij} are zero at the saddle point and, therefore, correspond to massive modes. The off-diagonal components, however, sit in a Mexican hat potential and have massive longitudinal fluctuations and massless transverse fluctuations. At the saddle point, $Q_{ij} \sim \epsilon_{ki}\partial_k\bar{\psi}\partial_j\psi$. The diagonal elements of Q_{ij} describe charge density fluctuations, $Q_{ii} \sim \epsilon_{ij}\partial_i\bar{\psi}\partial_j\psi$ [see Eq. (3)] and the off-diagonal elements describe exchange energy fluctuations, $\epsilon_{ij}Q_{ij} \sim \partial_i\bar{\psi}\partial_i\psi$. Let us define $Q_s = \frac{1}{2}\epsilon_{ij}Q_{ij}$, $Q_{c1} = Q_{11}$, and $Q_{c2} = Q_{22}$, so that $Q_{ij} = \epsilon_{ij}Q_s + \mathrm{diag}(Q_{c1}, Q_{c2})$. At the saddle point $Q_s = V\Lambda\bar{V}$ and $Q_c = 0$. Expanding the free energy functional, Eq. (8), to quadratic order in fluctuations of Q_{ij} about the saddle point, we find a typical diffusive effective action for the fluctuations, δQ_s, of Q_s about the saddle point. The only nonzero contributions to this action come from components of Q that are off diagonal in the advanced/retarded sector. Henceforth, we use Q to denote implicitly supermatrices with only these components nonzero. The corresponding actions for δQ_{c1} and δQ_{c2} contain massive propagators. It is important that one should not simply ignore these massive modes. They may lead to a renormalization of the diffusion constant for the massless modes [4,9]. Here, we find that this is not the case. The cross terms between δQ_s and $\delta Q_{c1}/\delta Q_{c2}$ are proportional to $|\mathbf{q}|^2$. Integrating out the massive modes induces a $|\mathbf{q}|^4$ term in the δQ_s propagator, which we neglect at small momentum. Keeping only fluctuations of Q_s over the saddle point manifold, we arrive at the sigma model

$$F[Q_s] = \int \frac{d^2 r}{32\rho_s}\mathrm{Str}\left[D_0|\nabla Q_s|^2 - 2i\left(\frac{\bar{\rho}\omega}{2}\right)\Lambda Q_s\right], \tag{9}$$

where D_0 is the classical diffusion constant given by $D_0 = \rho_s\tau_\Omega\Omega$. This energy functional describes the diffusive propagation of fluctuations in the exchange energy. All states are localized in this model [5,6], with a localization length

$$\xi = v_\Omega\tau_\Omega \exp\left[\frac{\pi^2 D_0^2}{64\rho_s^2}\right] \sim \Omega^{-3/2}\exp[\Omega^{-2}], \tag{10}$$

which is divergent in the Ω, $\mathbf{q} = 0$ limit. We have used $v_\Omega^2 = (dE/d\mathbf{p})^2|_{E=\Omega/2} = 2\rho_s\Omega/\bar{\rho}$. There is a crossover from an effectively delocalized regime at low frequency to a localized regime at high frequency. The divergence at $\mathbf{q} = 0$ is a natural consequence of the existence of an SU(2) global symmetry [broken to U(1) with the inclusion of the Zeeman coupling]. The Zeeman coupling is included by replacing Ω by $\Omega - gB$. There are no spin wave states below the Zeeman gap and the localization length diverges as the frequency approaches this, $\mathbf{q} = 0$, limit. Our task now is to determine whether this localization can be seen in experimental systems. Realistic correlations in the disorder potential, given by Eq. (4), lead to a modification of the parameters in the sigma model. The scattering time is given by [3]

$$\tau'_\Omega = \frac{4\bar{\rho}\rho_s^2}{\gamma'}\left(\frac{\bar{\rho}\Omega}{2}\right)^{-1}, \tag{11}$$

which again diverges as $\Omega \to 0$ such that $1/\tau_\Omega \ll \bar{\rho}\Omega/2$. When considering transport phenomena, the thermal lifetime given by Eq. (11) is renormalized by vertex corrections [3]. Scattering through an angle θ is weighted by a geometrical factor $\sin^2\theta$ and the resulting transport scattering time is a factor of 2 larger than the thermal scattering time. Subsequent calculations follow through as before, with a few additional complications. Finally, one obtains the same supersymmetric sigma model, Eq. (9), with

$$D'_0 = \rho_s \tau'_\Omega \Omega = \frac{16\rho_s^3}{\gamma'}. \tag{12}$$

The diffusion constant is now independent of Ω and the localization length no longer has an exponential dependence upon frequency, but a power law dependence:

$$\xi' \propto l_B\left(\frac{\Omega}{\rho_s}\right)^{-1/2}. \tag{13}$$

The numerical prefactor in this expression has an exponential dependence upon the disorder strength. Estimates assuming a donor density, n_d, of the same order as the electron density, $\bar{\rho}$, give prefactors upwards of 10^3. Localization lengths of a few thousand times the magnetic length are perfectly realistic at experimentally accessible frequencies.

The analysis presented above applies to noninteracting spin waves. However, the constraint $|\mathbf{n}|^2 = 1$ in Eq. (1) leads to interactions between spin waves when higher orders are included in the spin wave expansion (we call these nonlinear interactions). Coulomb interactions have also been neglected. One may ignore such interactions for the following reason: the effects of disorder are fundamentally single-particle effects. Nonlinear and Coulomb interactions are multiparticle effects. When the density of spin waves is very low, multiparticle effects may be neglected relative to single particle effects. Since spin waves are bosonic, thermally activated particles, this is achieved at low temperatures, $T \ll gB$. This is unlike fermionic systems, where one must consider the effect of interactions with a Fermi sea of particles. These considerations enter into the field-theoretical formulation via the frequency dependence of the interaction vertices. In a perturbative expansion, disorder-induced loops are integrated only over momentum, since the disorder scattering is elastic. Nonlinear and Coulomb interactions transfer energy between spin waves. Corresponding loops are integrated over momentum and summed over Matsubara frequencies. These frequency summations lead to factors of the Bose-Einstein distribution function which are exponentially small at low temperatures.

There are additional grounds upon which Coulomb interactions may be neglected relative to disorder interactions [10]. At sufficiently high temperature, one must include nonlinear interactions, but may ignore Coulomb interactions. This regime is beyond the scope of this Letter.

Experimental determination of the diffusion coefficient and localization length may be possible using space/time-resolved photoluminescence. In the QHF, magnetoexcitons and spin waves are identical [11]. If, in addition, the subband wave function of valence holes is centered in the same position as the wave function of electrons in the two-dimensional electron gas, then excitons have the same Hamiltonian as magnetoexcitons [12]. Localization and diffusion of excitons in other systems have been measured using photoemission spectroscopy [13]. The same techniques may be applicable here.

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