Holography, Cosmology, and the Second Law of Thermodynamics

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(Received 18 February 1999)

We propose that the holographic principle be replaced by the generalized second law of thermodynamics when applied to time-dependent backgrounds. For isotropic open and flat universes with a fixed equation of state, this agrees with the cosmological holographic principle proposed by Fischler and Susskind (hep-th/9806039). However, in more general situations, it does not. [S0031-9007(99)09398-9]

PACS numbers: 04.70.Dy, 05.70.Ce, 11.25.Mj, 98.80.Cq

The holographic principle states the maximum number of degrees of freedom in a volume should be proportional to the surface area [1,2]. This principle is based on earlier studies by Bekenstein [3] on maximum entropy bounds within a given volume. One argument used to motivate the holographic principle is as follows. Consider a region of space with volume V, bounded by an area A, which contains an entropy, S, and assume that this entropy is larger than that of a black hole with the same surface area. Now throw additional energy into this region to form a black hole. Assuming that the Bekenstein-Hawking formula, $S = A/4$, actually gives the entropy of the black hole, we conclude the generalized second law of thermodynamics [4] has been violated. (Note that the generalized second law is not related to the generalized entropy introduced by Tsallis [5] in a different context.) To avoid this contradiction, the holographic principle proposes that the entropy inside a given region must satisfy $S/A < 1$. However, this argument implicitly assumes that the black hole forms in a background that is otherwise static.

In the following, we examine how the argument changes in the more general time-dependent situations encountered in cosmology. We argue that the principle that replaces holography is simply that physics respects the generalized second law of thermodynamics [4]. For static backgrounds, this reduces to the holographic bound. However, more generally, a simple formula bounding the entropy inside a region by an amount proportional to the area does not hold.

Fischler and Susskind [6] have proposed a generalization of the holographic principle to certain cosmological backgrounds. This proposal has been studied further in [7,8]. Furthermore, in earlier work Bekenstein [9] examines a stronger entropy bound for Friedman-Robertson-Walker cosmologies. For flat and open universes with time-independent equations of state, we find that the Fischler-Susskind bound is in accord with the generalized second law. We propose a refinement of their bound that also applies to inflationary universes after reheating.

Fischler and Susskind found that closed universes violate their cosmological holographic bound and speculated that such backgrounds were either inconsistent or that new behavior sets in as the bound is violated. We argue that the evolution of closed universes does not violate the generalized second law, and hence such backgrounds are self-consistent.

A related problem we consider is how to apply the holographic principle in a volume inside the event horizon of a black hole. The naive holographic bound can easily be violated in such a region. At the same time the evolution is in accord with the generalized second law. It seems the price an observer pays for violating the holographic principle is to eventually encounter a curvature singularity. However, it is possible for this fate to be delayed for cosmological time scales.

I. The story so far.—Fischler and Susskind [6] realized that while the requirement that the holographic bound, $S/A < 1$, applies to an arbitrary region for the static case, the extension to cosmological spacetimes is more subtle. Specifically, the homogeneous energy density, ρ, of simple cosmological models implies a homogeneous entropy density, s. Inside a (comoving, spatial) volume $V \sim R^3$, the total entropy is $S = sV$. The boundary of this region has the physical area $A \sim a(t)^2R^2$, where $a(t)$ is the scale factor of the Robertson-Walker metric, so $S/A \sim sR/a(t)^2$. Consequently, for a fixed s it is always possible to choose a volume large enough to violate the holographic bound. Fischler and Susskind propose to resolve this problem by stipulating that the holographic bound applies only to regions smaller than the cosmological (particle) horizon [10], which corresponds to the forward light-cone of an event occurring at (or infinitesimally after) the initial singularity. The comoving distance to the horizon, r_H, is

$$r_h = \int_0^t \frac{1}{a(t')}\,dt' \tag{1}$$

while the corresponding physical distance is

$$d_h = a(t)r_h = a(t)\int_0^t \frac{1}{a(t')}\,dt'. \tag{2}$$

The scale factor obeys the evolution equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi}{3m_{\rm pl}^2}\rho - \frac{k}{a^2} \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\rm pl}}(\rho + 3p), \tag{4}$$

where k takes the values ± 1 and 0, for solutions with positive, negative, and zero spatial curvature. From here on we use natural units, where $m_{\rm pl} = \sqrt{8\pi}$.

For a perfect fluid, in a flat ($k = 0$) universe, whose pressure and density satisfy $\rho = \omega p$, the solution of Eqs. (3) and (4) is straightforward:

$$a(t) = a_0\left(\frac{t}{t_0}\right)^q, \qquad q = \frac{2}{3}\frac{1}{1+\omega}. \tag{5}$$

In particular, if $\omega = 0$ we recover the equation of state for dust, while $\omega = 1/3$ is the appropriate value for a hot (relativistic) gas or radiation. In general,

$$d_H = \frac{t}{1-q}. \tag{6}$$

The comoving entropy density is constant, so with $k = 0$ it follows that when measured over the horizon volume,

$$\frac{S}{A} \propto t^{1-3q}. \tag{7}$$

If $q < 1/3$ ($\omega > 1$) the holographic bound is violated at late times but, as Fischler and Susskind explain, such a cosmological model is not viable since a perfect fluid with $\omega > 1$ has a speed of sound greater than the speed of light.

In realistic cosmological models the equation of state is far from that of a perfect fluid with constant ω. Even simple models of the big bang combine dust and radiation and make a transition between $\omega = 1/3$ and $\omega = 0$, since the energy density of radiation drops faster than the density of dust as the universe expands. More importantly, during an inflationary epoch in the primordial universe, $\ddot{a}$ is, by definition, positive so the pressure and ω must be negative.

One of the original motivations for inflation was that it endows the primordial universe with a substantial entropy density. Inflationary models generate entropy after inflation has finished, when energy is transferred from the scalar field which drives the inflationary expansion to radiation and ultrarelativistic particles. This process is referred to as reheating, and the equation of state changes from $\omega < -1/3$, usually to a radiation dominated universe whose subsequent evolution is described by the "standard" model of the hot big bang. The comoving entropy density is not constant, and S/A is thus a more complicated function of time than it is in models with constant ω.

Rama and Sarkar [8] have discussed the application of the holographic bound to inflationary models. The maximum temperature, T, attained after inflation is model dependent, and the resulting entropy density is proportional to T^3 only if we assume a relativistic gas. Inflation makes the cosmological horizon arbitrarily large; for instance, it is not difficult for it to be 10^{1000} times greater than the value found in the absence of inflation. Applying the original Fischler-Susskind formulation of the holographic principle leads to a value of S/A massively greater than unity for almost any realistic inflationary model. This difficulty is noted by Rama and Sarkar, and they propose various smaller volumes over which to measure the entropy. In general, their formulation is not consistent with the one we propose in the next section.

II. Holography and the generalized second law.—One of the initial motivations for the holographic principle was based on the generalized second law of thermodynamics. The generalized second law states

$$\delta S_{\rm mat} + \delta S_{\rm BH} \geq 0, \tag{8}$$

where $S_{\rm mat}$ is the entropy of matter outside black holes, and $S_{\rm BH}$ is the Bekenstein-Hawking entropy of the black holes. This law has not been proven but is expected to follow from most of the current approaches to quantum gravity. There are many nontrivial situations where this law has been tested [3]. If we assume this law is correct, the holographic principle follows if we consider a region of space embedded in an approximately static background (such as Minkowski space, or anti–de Sitter space), as discussed in the introduction.

Our main interest is to study the formulation of holographic style bounds in time-dependent situations. Our guiding principle is the generalized second law of thermodynamics, rather than the holographic principle itself. Thus, the general principle which proposes to replace the holographic principle in time-dependent backgrounds is simply *the generalized second law of thermodynamics holds.*

In the examples considered below, we will assume changes are quasistatic. This implies that to an arbitrarily good approximation, the entropy is maximized, subject to constraints, at all times. In these situations we can make a stronger statement: *for all time the entropy is maximized subject to the constraints.*

For volumes embedded in certain backgrounds we may use these principles to deduce holographic style bounds on the entropy, but this does not appear to be possible in general. In the following examples we proceed by noting that while sufficiently large volumes of a nonstatic spacetime may have $S/A \gg 1$, the generalized second law will hold if these regions cannot (causally) collapse to form black holes while S/A is still much larger than unity. Therefore, the largest length scale over which we need to be concerned about S/A is the size of the (spatially) largest perturbation of the background spacetime which grows with time. If perturbations do not grow, black holes cannot form, $\delta S_{\rm BH} = 0$, and the generalized second law reduces to $\delta S_{\rm mat} \geq 0$, which is satisfied by any physically reasonable equation of state.

(A) Flat universe.—Let us consider isotropic, homogeneous, and spatially flat cosmologies. The comoving entropy density in these models is constant. This is consistent with the generalized second law. If $S/A < 1$ initially, this condition is satisfied at all later times, provided ω is fixed and less than unity.

It remains to be seen whether $S/A > 1$ at earlier times, and if this would imply a violation of the generalized second law. In order to discuss this, we need to introduce a length scale. Rather than the particle horizon, which is arbitrarily large in inflationary models, we focus on the Hubble length, or Hubble horizon size, H^{-1}. Physically, H^{-1} is the distance at which a point appears to be receding at the speed of light due to the overall expansion of the universe. More relevantly, gravitational perturbation theory shows that small perturbations to a spatially flat, homogeneous, and isotropic universe with wavelengths larger than H^{-1} do not grow with time [11]. While it is possible that $S/A > 1$ in a volume much larger than H^{-3}, this region is unable to collapse to form a black hole and violate the second law. Thus the maximum volume over which we need to be concerned that S/A is the Hubble volume, H^{-3}.

If ω is constant the particle horizon, d_H, and H^{-1} are related to one another by a factor of order unity, and we recover the Fischler-Susskind formulation. However, if inflation has taken place the particle horizon is much larger than H^{-1}, which depends only on the instantaneous expansion rate and not on the integrated history of the universe.

As an example, consider the energy density and entropy density for a relativistic gas at temperature T:

$$\rho = \frac{\pi^2}{30} n_* T^4, \tag{9}$$

$$s = \frac{2\pi^2}{45} n_* T^3, \tag{10}$$

where n_* is the number of bosonic degrees of freedom plus $7/8$ times the number of fermionic degrees of freedom. Using Eq. (3) to relate ρ and H,

$$\frac{S}{A} \leq \sqrt{n_*}\, T\,, \tag{11}$$

up to constant factors. Since the density must be less than unity for quantum gravitational corrections to be safely ignored, the maximum temperature is proportional to $n_*^{-1/4}$, and the maximum value of S/A inside a Hubble volume is proportional to $n_*^{1/4}$. Violating $S/A < 1$ significantly at a sub-Planckian energy requires an enormous value of n_*, which constitutes a fine-tuning.

Furthermore, a perturbation the size of the Hubble horizon does not collapse into a black hole instantaneously. Thus the bound would need to be violated during the time it took the black hole to form, and S/A decreases with time. Thus, in the absence of tuning the holographic bound we have proposed is satisfied at all post-Planckian times.

(B) Closed universe.—For the case of an isotropic closed universe with fixed equation of state, Fischler and Susskind found [6] that even if $S/A < 1$ initially on particle horizon sized regions, it could be violated at later times. This violation is possible even while the universe is still in its expansion phase.

The generalized second law, on the other hand, is expected to hold on particle horizon sized regions in a closed universe. For the sake of definiteness, suppose the violation occurs while the universe is still in its expansion phase. One would certainly expect that a region with an excessive entropy density could start to collapse via the Jeans instability and form a black hole. However, simple estimates indicate the time taken to form such a black hole is of the order of the lifetime of the closed universe. Thus while it is conceivable that a region with $S/A > 1$ could form a black hole in a closed universe, the process of its formation would require a time comparable to the overall lifetime of the universe which contains it. Consequently, a violation of the holographic bound of Fischler and Susskind remains consistent with the second law for cosmologically long time scales.

Moreover, to find $S/A > 1$ well before the closed universe reaches its final singularity we must consider a volume that is a substantial fraction of the overall universe. The collapse of this region into a single black hole is not a small perturbation of the background Friedmann universe, and the naive use of the evolution equations for the unperturbed universe to discuss the entropy density of collapsing region is an assumption of dubious validity.

(C) Open universe.—The behavior of isotropic open (with negative spatial curvature) universes is similar to that of flat universes. If $S/A < 1$ initially, it remains so at later times [6]. An argument that $S/A < 1$ remains valid at earlier times can likewise be made in a similar way to the flat case.

Fischler and Susskind also considered the case of certain anisotropic flat universes. In these cases it was found S/A was constant in time.

(D) Inside a black hole.—Another time-dependent background of interest is the region inside the event horizon of a black hole. If we consider a spatial volume inside the event horizon, the generalized second law will apply if the volume is out of thermal contact with other regions. However, it is straightforward to argue that the entropy in such a volume, with size of order the horizon size, can exceed its surface area. In fact, the physics inside a black hole should closely resemble that of the closed universe, at sufficiently late times.

We see no reason why an observer inside such a region should not be able to actually measure a violation of the holographic bound. A direct measurement is difficult since the observer will typically hit the singularity within a light-crossing time of the black hole horizon. However, if the observer has the additional information that the entropy density is constant, he/she can infer a violation of holography via local measurements.

(E) Inflating universe.—We can view an inflationary universe as a Friedmann universe with a time-dependent equation of state. During the reheating phase at the end of inflation there is a sharp change in the equation of state, as energy is transferred from the inflation field to radiation (or ultrarelativistic particles). This raises the entropy of the universe in a homogeneous way. After this sudden increase in the entropy density it is possible to violate Fischler and Susskind's bound when it is applied to regions the size of the particle horizon. Of course, a sharp homogeneous increase in the entropy density is permitted by the generalized second law.

The process of reheating is model dependent. To simplify the discussion assume that reheating takes place instantaneously. After reheating, the postinflationary universe resembles a universe which never inflated, the only difference being the much larger particle horizon in an inflationary universe. We may therefore adopt the results for Friedmann universes with a constant equation of state. The postinflationary universe is accurately approximated by a flat, isotropic spacetime, so if $S/A < 1$ when measured over a Hubble volume at the end of inflation, this inequality will continue to be satisfied at later times. Moreover, immediately after reheating the energy density is typically well below the Planck scale so $S/A \ll 1$ in the absence of extreme fine-tuning. This bound differs from that of Rama and Sarkar [8], and we obtain no specific constraints on inflationary models beyond the usual assumption that the energy density is sub-Planckian during and after inflation.

III. Conclusions.—We have proposed that the holographic principle be replaced by the generalized second law of thermodynamics [4] for general time-dependent backgrounds. In static backgrounds, this reduces to the holographic principle of Susskind and of 't Hooft [1,2]. For cosmological backgrounds, corresponding to spatially open universes with fixed equations of state, the second law implies the entropy bound of Fischler and Susskind [6] on particle horizon sized regions. However, for closed universes, and inside black hole event horizons, a useful holographic bound does not follow from the second law. For inflationary universes, we show that in the absence of fine-tuning $S/A \ll 1$ in any region of the postinflationary universe that can undergo gravitational collapse. This ensures that the generalized second law holds, but the formulation of the cosmological holographic principle given by Fischler and Susskind requires modification.

This work was supported in part by DOE Grant No. DE-FE0291ER40688-Task A.

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