Statistical Entropy of Four-Dimensional Rotating Black Holes from Near-Horizon Geometry

Mirjam Cvetič[1,2] and Finn Larsen[1]
[1]*David Rittenhouse Laboratories, University of Pennsylvania, Philadelphia, Pennsylvania 19104*
[2]*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106*
(Received 27 May 1998)

We show that a class of four-dimensional rotating black holes allow five-dimensional embeddings as black rotating strings. Their near-horizon geometry factorizes locally as a product of the three-dimensional anti–de Sitter space-time and a two-dimensional sphere ($AdS_3 \times S^2$), with angular momentum encoded in the global space-time structure. Following the observation that the isometries on the AdS_3 space induce a two-dimensional (super)conformal field theory on the boundary, we reproduce the microscopic entropy with the correct dependence on the black hole angular momentum.
[S0031-9007(98)08229-5]

PACS numbers: 04.70.Dy, 11.25.Mj

Recent developments in nonperturbative string theory have provided a fruitful framework to consider quantum properties of black holes. In particular, extreme black holes with Ramond-Ramond (RR) charges can be interpreted in higher dimensions as intersecting D-branes [1], and this has led to a counting of black hole quantum states that agrees precisely with the Bekenstein-Hawking (BH) entropy [2]. This counting is carried out in the weakly coupled regime where the D-brane constituents of the black hole experience *flat* space-time geometry; however, due to supersymmetry, it remains valid in the regime where the D-branes are strongly coupled, and the geometric space-time description of black holes emerges. Thus the microscopic derivation of the BH entropy is justified, but it is difficult to explore the quantum black hole geometry in detail using D-branes.

This obstacle was recently overcome when Strominger [3] (and also Birmingham *et al.* [4]) proposed a new method that counts the black hole microstates directly using the geometry of the black hole. The central observation is that, when embedded in a higher dimensional space, the near-horizon geometry locally contains the three-dimensional anti–de Sitter space-time (AdS_3). The microscopic theory of such backgrounds must realize a two-dimensional conformal field theory (CFT) and its central charge can be computed from general principles [5]. The quantum degeneracy of the theory follows from these facts and a few additional mild assumptions; and the remarkable observation of [3] is that it accounts correctly for the black hole entropy. Although this result is logically independent of the D-brane countings, it has nevertheless been conjectured that the theory inferred from the AdS_3 is equivalent to the D-brane theory [6].

Shortly after the original D-brane counting of static black hole states, it was realized that the D-brane method also accounts for the entropy of rotating black holes [7,8]. The appropriate states belong to the same CFT that accounts for the entropy of nonrotating black holes; they simply have different quantum numbers.

The purpose of this Letter is to include the effects of angular momentum within the new geometric approach. It appears at first sight that this goal is doomed to failure: rotating black holes are flattened relative to their nonrotating brethren, and they couple the angular and temporal components of the metric. These effects seem to preclude the appearance of an AdS_3 space-time in the near-horizon geometry, thus rendering the new method inapplicable, and furthermore showing that the "geometric CFT" and the "D-brane CFT" are distinct. However, we shall demonstrate that this intuition is incorrect: the near-horizon geometry of rotating black holes does contain an AdS_3 component, and it reproduces the black hole entropy correctly.

We consider a family of four-dimensional near-extreme rotating black holes and interpret these black holes as charged rotating strings in five dimensions, with the string wrapped around the compactified fifth dimension. Its near-horizon geometry is found to be *locally* a direct product of the AdS_3 and a two-dimensional sphere ($AdS_3 \times S^2$); the sphere is not squashed by the rotation. (For the correct microscopic interpretation it is essential that the solution is reinterpreted in five dimensions as a rotating string; in four dimensions the near-horizon geometry does not have a factorized form.) Instead the rotation is implemented by boundary conditions that couple the angular and the temporal components of the metric, with the rotation absent in the "comoving coordinate system." Related work on five-dimensional rotating black holes is presented in detail elsewhere [9].

The starting point is a large class of four-dimensional black holes (of toroidally compactified string theory), whose explicit space-time metric is given in [10]. They are specified by their mass M, four U(1) charges Q_i, and the angular momentum J or, more conveniently, in terms of the nonextremality parameter m, four boosts δ_i, and the angular parameter l,

 0031-9007/99/82(3)/484(4)$15.00

$$G_4 M = \frac{1}{4}\, m \sum_{i=0}^{3} \cosh 2\delta_i \,,$$

$$G_4 Q_i = \frac{1}{4}\, m \sinh 2\delta_i ; \qquad i = 0,1,2,3 \,,$$

$$G_4 J = ml\left(\prod_{i=0}^{3} \cosh \delta_i - \prod_{i=0}^{3} \sinh \delta_i\right),$$

where G_4 is the four-dimensional Newton's constant. (The notation follows [10]. The r_0 of [11] is $r_0 = 2m$, the μ of [12] is $m = 4\mu$, the l of [12] is $l_{\mathrm{here}} = 4 l_{\mathrm{there}}$, and the Q_i of [13] is $Q_{\mathrm{here}} = 2 Q_{\mathrm{there}}$.) The Kerr-Newman black hole corresponds to the case where the four charges are identical. The extreme limit is obtained by taking $m \to 0$ and $l \to 0$ while keeping $Q_{0,1,2,3}$ finite; in this case $J = 0$. From the explicit solution one finds the BH entropy [10],

$$S \equiv \frac{A_4}{4G_4} = \frac{\pi}{4G_4}\left[8m^2\left(\prod_{i=0}^{3} \cosh \delta_i + \prod_{i=0}^{3} \sinh \delta_i\right) + 8m\sqrt{m^2 - l^2}\left(\prod_{i=0}^{3} \cosh \delta_i - \prod_{i=0}^{3} \sinh \delta_i\right)\right], \quad (1)$$

where A_4 is the area of the outer horizon.

A specific representation of the metric and its accompanying matter fields is given in [10] in terms of the NS-NS fields (NS: Neveu-Schwarz); e.g., its higher-dimensional interpretation is that of a rotating fundamental string with winding and momentum modes, superimposed with the Kaluza-Klein monopole and the H monopole [14]. A particular duality transformation leaves the four-dimensional space-time invariant, while mapping this configuration to three intersecting M5-branes of M theory (specified by $Q_{1,2,3}$), with momentum (specified by Q_0) along the common string. This M-theory configuration can be interpreted as a rotating string in five dimensions after toroidal compactification. The space-time metric of the rotating string is rather complicated, and we were unable to write it in a relatively compact form. [The complications associated with the angular momentum are similar to those of adding an additional charge (the "fifth parameter") to the configuration [15].] However, the metric simplifies significantly in the near-horizon region $r \ll Q_{1,2,3}$, when the condition $\delta_{1,2,3} \gg 1$ is satisfied. Then the metric of the five-dimensional rotating string in the Einstein frame becomes

$$ds_5^2 = \frac{2}{\lambda}\left[\left(r - \frac{l^2}{2m}\cos^2\theta\right)(-d\tilde{t}^2 + d\tilde{y}^2) + 2m\left(1 - \frac{l^2}{2m^2}\right)\cos^2\theta\, d\tilde{t}^2 - \frac{l^2}{m}\cos^2\theta\, d\tilde{t}\, d\tilde{y}\right] + \frac{\lambda^2}{4}\left[\frac{1}{r^2 - 2mr + l^2}\, dr^2 + d\theta^2 + \sin^2\theta\, d\phi^2\right] - \sqrt{\frac{\lambda l^2}{m}}\,(d\tilde{y} + d\tilde{t})\sin^2\theta\, d\phi \,,$$

where the boosted variables (specifying the momentum along the string) are

$$d\tilde{t} = \cosh\delta_0 dt - \sinh\delta_0 dy, \qquad d\tilde{y} = \cosh\delta_0 dy - \sinh\delta_0 dt\,, \quad (2)$$

and the characteristic length scale λ is defined as $\lambda \equiv 8G_4(Q_1 Q_2 Q_3)^{1/3}$. Note that the metric (2) retains nontrivial dependence on the angular momentum; however, the Kerr-Newman black holes are *not* compatible with the limit considered here.

Introducing the shifted coordinate,

$$d\tilde{\phi} = d\phi - \frac{2l}{\sqrt{\lambda^3 m}}\,(d\tilde{y} + d\tilde{t})\,, \quad (3)$$

yields the factorized metric

$$ds_5^2 = \frac{2}{\lambda}\left[-\left(r - 2m + \frac{l^2}{2m}\right)d\tilde{t}^2 - \frac{l^2}{m}\, d\tilde{t}\, d\tilde{y} + \left(r - \frac{l^2}{2m}\right)d\tilde{y}^2\right] + \frac{\lambda^2}{4}\left[\frac{dr^2}{r^2 - 2mr + l^2} + d\theta^2 + \sin^2\theta\, d\tilde{\phi}^2\right].$$

With this choice of coordinates it is apparent that the geometry is a direct product of a two-sphere S^2, with radius $\frac{\lambda}{2}$, and a Banados, Teitelboim, and Zanelli (BTZ) black hole in three space-time dimensions with a negative cosmological constant $\Lambda = -\lambda^2$. Indeed, introducing the coordinates $\tau \equiv \frac{t\lambda}{R_{11}}$, $\sigma \equiv \frac{y}{R_{11}}$, and $\rho^2 \equiv \frac{2R_{11}^2}{\lambda}[r + 2m\sinh^2\delta_0 - \frac{l^2}{2m}(\cosh\delta_0 - \sinh\delta_0)^2]$, where R_{11} is the radius of the dimension wrapped by the string, we find the standard BTZ metric [16]

$$ds_5^2 = -N^2 d\tau^2 + N^{-2} d\rho^2 + \rho^2(d\sigma - N_\sigma d\tau)^2 + \frac{1}{4}\lambda^2 d\tilde{\Omega}_2^2\,,$$

$$N^2 = \frac{\rho^2}{\lambda^2} - M_3 + \frac{16G_3^2 J_3^2}{\rho^2}\,, \qquad N_\sigma = \frac{4G_3 J_3}{\rho^2}\,,$$

where the effective BTZ mass M_3 and angular momentum J_3 are

$$M_3 = \frac{R_{11}^2}{\lambda^3}\left[\left(4m - \frac{2l^2}{m}\right)\cosh 2\delta_0 + \frac{2l^2}{m}\sinh 2\delta_0\right],$$

$$8G_3 J_3 = \frac{R_{11}^2}{\lambda^2}\left[\frac{2l^2}{m}\cosh 2\delta_0 + \left(4m - \frac{2l^2}{m}\right)\sinh 2\delta_0\right],$$

and the effective three-dimensional gravitational coupling G_3 is related to the four-dimensional one G_4 as [13]

$$\frac{1}{G_3} = \frac{1}{G_4}\,\frac{A_2}{2\pi R_{11}}\,, \quad (4)$$

where $A_2 = \pi\lambda^2$ is the area of the two-sphere S^2. The BTZ geometry is *locally* AdS_3 but global identifications ensure causal structures that are similar to those familiar from four-dimensional black holes. For our purposes it is crucial that the BTZ geometry is *asymptotically* AdS_3, because then the isometries induce a CFT on the boundary at spatial infinity [3,5]. Its central charge c is determined by the effective cosmological constant $-\lambda^2$ as [5]

$$c = \frac{3\lambda}{2G_3} = 6\,\frac{Q_1Q_2Q_3}{8G_4R_{11}}\,, \tag{5}$$

and the conformal weights $h_{L,R}$ (eigenvalues of the Virasoro operators L_0, $\bar{L}_0$, respectively) are related to the BTZ parameters as [17]

$$h_{L,R} = \frac{\lambda M_3 \pm 8G_3J_3}{16G_3}\,. \tag{6}$$

The shift Eq. (3) introduces a coupling between the AdS_3 and the S_2, but Eqs. (5) and (6) are still justified because their derivations apply at each point on the S_2.

Collecting the formulas (5) and (6) we find, in the semiclassical regime where the conformal weights are large, the statistical entropy

$$\begin{aligned} S &= 2\pi\left(\sqrt{\frac{c}{6}\,h_L} + \sqrt{\frac{c}{6}\,h_R}\right) \\ &= \frac{\pi}{4G_4}\sqrt{Q_1Q_2Q_3}\left[\sqrt{m}\,e^{\delta_0} + \sqrt{m - \frac{l^2}{m}}\,e^{-\delta_0}\right]. \end{aligned} \tag{7}$$

On the other hand, we assume parameters satisfying $\delta_{1,2,3} \gg 1$ and so the BH entropy (1) becomes

$$S \simeq \frac{\pi}{4G_4}\sqrt{Q_1Q_2Q_3}\left[\sqrt{m}\,e^{\delta_0} + \sqrt{m - \frac{l^2}{m}}\,e^{-\delta_0}\right]. \tag{8}$$

This is in precise agreement with the microscopic calculation (7). It also agrees with the D-brane motivated counting given in [11].

The derivation of statistical black hole entropy does not rely on the details of the underlying quantum theory, but the relation to M theory is interesting. In M-theory units $R_{11} = g\sqrt{\alpha'}$, the Planck length is $l_p = (2\pi g)^{1/3}\sqrt{\alpha'}$, and $G_4 = \frac{1}{8}\,\frac{(\alpha')^4g^2}{R_1R_2R_3R_4R_5R_6}$ where the R_i are the radii of the compact dimensions and g is the string coupling constant.

In the preceding section we assumed the near-horizon approximation $r \ll Q_{1,2,3}$ and the condition $\delta_{1,2,3} \gg 1$. These become exact in the formal decoupling limit [6],

$$\begin{aligned} &(l_p, r, m, l) \to 0, \\ &\quad \text{with } (r \sim l_p^3, m \sim l_p^3, l \sim l_p^3, R_{1,\ldots,6} \sim l_p, R_{11} \sim 1)\,, \end{aligned} \tag{9}$$

where the field theory on the intersection of the M5-branes decouples from gravity. Note, in particular, that angular momentum is compatible with decoupling. This appears to be the case only for configurations that correspond to regular black holes in four and five dimensions; the near-horizon geometry of, e.g., the (coincident) D3-branes, and the M5-branes do not have rotating versions. Thus only the induced CFTs in *two* dimensions seems to have world-volume currents with charges that can be interpreted as angular momenta.

The quantization conditions on the D-brane charges are [1] $Q_i = [(\sqrt{\alpha'})^3/R_{2i-1}R_{2i}]n_i g$, where $n_{1,2,3}$ is the number of coincident M5-branes with a given orientation, so $Q_1Q_2Q_3 = 8G_4R_{11}n_1n_2n_3$, and from (5) the quantized form of the central charge becomes $c = 6n_1n_2n_3$ as expected [18–20]. A heuristic microscopic interpretation of this formula is that each of the M-branes traverse the intersection string n_i times, giving a total of $n_1n_2n_3$ distinct topological sectors, each associated with 6 degrees of freedom.

The quantum numbers ϵ and p for the string energy and momentum, respectively, are introduced through

$$\begin{aligned} E &= 2m\cosh 2\delta_0 = 8G_4\,\frac{\epsilon}{R_{11}}\,, \\ Q_0 &= 2m\sinh 2\delta_0 = 8G_4\,\frac{p}{R_{11}}\,, \end{aligned} \tag{10}$$

and then the conformal weights $h_{L,R}$ can be written as

$$\begin{aligned} h_L &= \frac{R_{11}}{8G_4}\,me^{2\delta_0} = \frac{1}{2}(\epsilon + p), \\ h_R &= \frac{R_{11}}{8G_4}\left(m - \frac{l^2}{m}\right)e^{-2\delta_0} = \frac{1}{2}(\epsilon - p) - \frac{1}{n_1n_2n_3}J^2. \end{aligned} \tag{11}$$

The space-time angular momentum is normalized so that J is measured in units of $\hbar$, and thus, from semiclassical reasoning, J is quantized as an integer. By introducing a single unit of angular momentum we see that h_R is quantized in units of $1/n_1n_2n_3$.

The angular momentum of the black hole breaks rotational invariance of the background, so it is not guaranteed by symmetries that the near-horizon geometry contains a two-sphere S^2. In the present model the linking of AdS_3 and S^2 is accomplished by the *global* features contained in the boundary conditions at infinity and encoded in the coordinate shift (3). It is therefore surprisingly simple to include angular momentum while preserving full analytical control. This makes the present model an attractive setting to study angular momentum. The precise value of the shift can be understood as follows: the potentials conjugate to the left- and right-moving string energies are

$$\beta_L = \frac{\pi}{2}\sqrt{\frac{\lambda^3}{m}}\,e^{-\delta_0}, \qquad \beta_R = \frac{\pi}{2}\sqrt{\frac{\lambda^3m}{m^2 - l^2}}\,e^{\delta_0}, \tag{12}$$

respectively, and the rotational velocity Ω is given through $\beta_H\Omega = (2\pi l/\sqrt{m^2 - l^2})$, where $\beta_H = \frac{1}{2}(\beta_L + \beta_R)$ is the inverse of the Hawking temperature. Thus, in the "comoving" frame where the $\tilde{\phi}$, given in (3), is fixed, we have

$$\left(\frac{d\phi}{dt}\right)_{t=y,\tilde{\phi}} = \frac{4l}{\sqrt{\lambda^3 m}}\, e^{-\delta_0} = \frac{\beta_H \Omega}{\beta_R}\,, \qquad (13)$$

so the azimuthal angle ϕ is essentially shifted by the angular velocity Ω. The factors of inverse temperatures and their significance for the wave functions of black hole perturbations are similar to the ones discussed for five-dimensional black holes in [9].

The direct connection between the near-horizon geometry and the underlying CFT appears to be valid for black holes in the near-extreme limit only. Eventually, it will be important to test its validity and limitations away from the near-extreme limit. The structure indicated by angular momentum may play an important role in this endeavour [10,21].

M. C. thanks J. Polchinski and other participants in the duality program at ITP, Santa Barbara, for discussions. This work is supported in part by DOE Grant No. DOE-FG02-95ER40893. The work at the ITP was further supported by the NSF under Grant No. PHY94-07194.

[1] J. Polchinski, Phys. Rev. Lett. **75**, 4724–4727 (1995).
[2] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99–104 (1996).
[3] A. Strominger, J. High Energy Phys. **2**, 9 (1998).
[4] D. Birmingham, I. Sachs, and S. Sen, Phys. Lett. B **424**, 275–280 (1998).
[5] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207–226 (1986).
[6] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231–252 (1998).
[7] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa, Phys. Lett. B **391**, 9602065 (1997).
[8] J. Breckenridge, D. Lowe, R. Myers, A. Peet, A. Strominger, and C. Vafa, Phys. Lett. B **381**, 423–426 (1996).
[9] M. Cvetič and F. Larsen, Nucl. Phys. **B531**, 239–255 (1998).
[10] M. Cvetič and D. Youm, Phys. Rev. D **54**, 2612–2620 (1996).
[11] G. T. Horowitz, D. A. Lowe, and J. M. Maldacena, Phys. Rev. Lett. **77**, 430–433 (1996).
[12] M. Cvetič and F. Larsen, Nucl. Phys. **B506**, 107–120 (1997).
[13] V. Balasubramanian and F. Larsen, Nucl. Phys. **B528**, 229–237 (1998).
[14] M. Cvetič and A. Tseytlin, Phys. Lett. B **366**, 95–103 (1996).
[15] M. Cvetič and D. Youm, Nucl. Phys. **B472**, 249–267 (1996).
[16] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[17] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506–1525 (1993).
[18] A. Strominger and J. Maldacena, Phys. Rev. Lett. **77**, 428–429 (1996).
[19] I. Klebanov and A. Tseytlin, Nucl. Phys. **B475**, 179–192 (1996).
[20] V. Balasubramanian and F. Larsen, Nucl. Phys. **B478**, 199 (1996).
[21] F. Larsen, Phys. Rev. D **56**, 1005–1008 (1997).