Energy of a Plasma in the Classical Limit

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When $\lambda_T \ll d_T$, where λ_T is the de Broglie wavelength and d_T is the distance of closest approach of thermal electrons, a classical analysis of the energy of a plasma can be made. In all the classical analysis made until now, it was assumed that the frequency of the fluctuations $\omega \ll T$, $(k_B = \hbar = 1)$. Using the *fluctuation-dissipation theorem,* we evaluate the energy of a plasma, allowing the frequency of the fluctuations to be arbitrary. We find that the energy density is appreciably larger than previously thought for many interesting plasmas, such as the plasma of the Universe before the recombination era. [S0031-9007(99)09376-X]

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There have been many classical calculations of the energy of a plasma [1–3]. They are based on perturbation theory of an ideal gas, in terms of the plasma parameter g (which usually is a small value). The treatment, to the first order in g, is the Debye-Hückel theory. However, in the calculations that have been made it is assumed that $\omega \ll T$ $(k_B = \hbar = 1)$. This is a very strong assumption. For example, in our previous analysis [4,5], we showed that only by not assuming $\omega \ll T$, is the blackbody spectrum obtained.

We evaluate the energy of a plasma, studying the electromagnetic fluctuations in a plasma without assuming that $\omega \ll T$. A plasma in thermal equilibrium sustains fluctuations of the magnetic and electric fields. The electromagnetic fluctuations are described by the fluctuation-dissipation theorem [6].

The evaluation of the electromagnetic fluctuations in a plasma has been made in numerous studies [7]. Recently, Cable and Tajima [8] (see also [9]) studied the magnetic field fluctuations in a cold plasma description with a constant collision frequency as well as for a warm, gaseous plasma, described by kinetic theory.

Using a model that extends the work of Cable and Tajima [8], we study an electron-proton plasma of temperature 10^4–10^5 K with densities 10^{13}–10^{19} cm^{-3}. The condition for a classical analysis is that $\lambda_T < d_T$, where λ_T is the de Broglie wavelength for a thermal electron and $d_T = e^2/T$, the distance of closest approach. This condition is satisfied for $T < 3 \times 10^5$ K and for the plasmas studied.

In Sec. I we recall the expressions for the electromagnetic fluctuations in a plasma, and in Sec. II, the electromagnetic energy is computed. Finally, we discuss our results in Sec. III.

(I) Electromagnetic fluctuations.—The spectra of the electromagnetic fluctuations in an isotropic plasma are given by [6]

$$\frac{\langle E^2\rangle_{\mathbf{k}\omega}}{8\pi} = \frac{1}{e^{\omega/T} - 1}\frac{\mathrm{Im}\varepsilon_L}{|\varepsilon_L|^2} + 2\frac{1}{e^{\omega/T} - 1}\frac{\mathrm{Im}\varepsilon_T}{|\varepsilon_T - (\frac{kc}{\omega})^2|^2}, \qquad (1)$$

$$\frac{\langle B^2\rangle_{\mathbf{k}\omega}}{8\pi} = 2\frac{1}{e^{\omega/T} - 1}\left(\frac{kc}{\omega}\right)^2\frac{\mathrm{Im}\varepsilon_T}{|\varepsilon_T - (\frac{kc}{\omega})^2|^2} \qquad (2)$$

$(\hbar = k_B = 1)$, where ε_L and ε_T are, respectively, the longitudinal and transverse dielectric permittivities of the plasma. The first and second terms in Eq. (1) are the longitudinal and transverse electric field fluctuations, respectively.

By using the fluctuation-dissipation theorem, we can estimate the energy in the electromagnetic fluctuations for all frequencies and wave numbers. The calculation includes not only the energy of the fluctuations in the well defined modes of the plasma, such as plasmons in the longitudinal component and photons in the transverse component, but also the energy in fluctuations that do not propagate.

For the description of the plasma, we use the model described in detail in Opher and Opher [4,5]. The description includes thermal and collisional effects. It uses the equation of Vlasov in first order, with the BGK (Bhatnagar-Gross-Krook) collision term that is a model equation of the Boltzmann collision term [10]. We used the BGK collision term as a rough guide for the inclusion of collisions in a plasma.

From this description, the dielectric permittivities for an isotropic plasma are easily obtained:

$$\varepsilon_L(\omega,\mathbf{k}) = 1 + \sum_\alpha \frac{\omega_{p\alpha}{}^2}{k^2 v_\alpha{}^2}\frac{1 + [(\omega + i\eta)/\sqrt{2}kv_\alpha]Z[(\omega + i\eta_\alpha)/\sqrt{2}kv_\alpha]}{1 + (i\eta/\sqrt{2}kv_\alpha)Z[(\omega + i\eta_\alpha)/\sqrt{2}kv_\alpha]}, \qquad (3)$$

$$\varepsilon_T(\omega,\mathbf{k}) = 1 + \sum_\alpha \frac{\omega_{p\alpha}{}^2}{\omega^2}\left(\frac{\omega}{\sqrt{2}kv_\alpha}\right)Z\left(\frac{\omega + i\eta_\alpha}{\sqrt{2}kv_\alpha}\right), \qquad (4)$$

where α is the label for the species of particles, v_α is the thermal velocity for the species, and $Z(z)$ is the Fried and Conte function.

(II) Electromagnetic energy.—In order to estimate the electromagnetic energy, we use the dielectric permittivities, given by Eqs. (3) and (4), and calculate the magnetic and the electric field spectra from Eqs. (1) and (2). Integrating the spectra in wave number and frequency [and dividing by $(2\pi)^3$], we obtain the energy densities of the magnetic field ρ_B and of the transverse and longitudinal electric fields ρ_{E_T} and ρ_L.

Usually, when estimating the energy stored in the electromagnetic fluctuations from Eqs. (1) and (2), it is assumed that $\omega \ll T$ ($k_B = \hbar = 1$). With this assumption, the Kramers-Kronig relations can then be used, and a simple expression for the energy is obtained [1,2]. However, the assumption that $\omega \ll T$ is very restrictive. For example, a large part of the fluctuations which create the blackbody electromagnetic spectrum has $\omega > T$ [4,5]. It is therefore necessary to perform the integration of the spectra over frequency and wave number without using this assumption.

Our model uses kinetic theory with a collision term that describes the binary collisions in the plasma. A cutoff has to be taken since, for very small distances, the energy of the Coulomb interaction exceeds the kinetic energy. This occurs for distances $r_{\min} \sim e^2/T$, which defines our maximum wave number, $k_{\max}$.

A large $k_{\max}$ is needed in order to reproduce the blackbody spectrum. In this study, we used a $k_{\max}$ equal to the inverse of the distance of closest approach, which we previously found is able to do this [4,5]. Any smaller $k_{\max}$ was unable to reproduce the entire blackbody spectrum.

In the usual classical calculations, the correction to the energy due to correlations between the particles is made through the *correlation energy*. To the first order in the plasma parameter g, the correlation energy depends on the two-particle correlation function $S(k)$,

$$E_C = \frac{n}{4\pi^2}\int dk\, k^2\phi_k S(k) - \frac{n}{4\pi^2}\int dk\, k^2\phi_k\,, \quad (5)$$

where the second term is the energy of the particles due to their own fields. $S(k)$ can be estimated by the fluctuation-dissipation theorem or by the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy equations [3]. Generally, it is assumed that $\omega \ll T$ (so the Kramers-Kronig relation can be used) and $S(k)$ is obtained as

$$S(k) = \frac{k^2}{k^2 + k_D^2}\,, \quad (6)$$

where k_D is the inverse of the Debye length.

With this, the energy density of a plasma to first order in g is given as

$$U = \frac{3}{2}\, nT\left[1 - \left(\frac{g}{12\pi}\right)\right], \quad (7)$$

where n is the number density of the particles. Thus, the correlation energy to the first order in g is

$$E_c = -\frac{3}{2}\, nT\left(\frac{g}{12\pi}\right). \quad (8)$$

We define the energy of a plasma as

$$U = \frac{3}{2}\, nT(1 + \Delta)\,. \quad (9)$$

With this definition, $\Delta = \Delta_0 = -g/12\pi$ for the previous classical analysis [Eq. (7)], where the subscript "0" means that the assumption $\omega \ll T$ has been used.

Higher order calculations of the correlation energy have been made, for example, by O'Neil and Rostoker [11]. However, in all treatments, the assumption $\omega \ll T$ has been made. As we commented above, the assumption $\omega \ll T$ is very strong. A large part of the fluctuations has $\omega > T$.

To obtain the interaction energy, we need to subtract the energy of the particles due to their own fields, the second term of Eq. (5), from the longitudinal energy density, ρ_L. We thus have $\rho_{\rm int} = \rho_L - \frac{n}{4\pi^2}\int dk\, k^2\phi_k$. Using Eq. (9), the interaction energy can be written as $\rho_{\rm int} \equiv \frac{3}{2}(nT)\Delta$, where $\rho_{\rm int}$ is the equivalent of the correlation energy. In fact, using the approximation $\omega \ll T$, $\rho_{\rm int}$ is equal to the second term of Eq. (7).

In order to compare $\rho_{\rm int}$ with E_c, we define the parameter,

$$F \equiv \frac{|\Delta| - |\Delta_0|}{|\Delta_0|}\,. \quad (10)$$

We previously found [5] that the transverse energy (summing the transverse electric and magnetic field energies, ρ_{E_T} and ρ_B) has an additional energy, compared to the blackbody energy density in vacuum. The additional transverse energy is

$$\Delta\rho_\gamma = \rho_B + \rho_{E_T} - \rho_\gamma\,, \quad (11)$$

where ρ_γ is the photon energy density, estimated as the blackbody energy density in vacuum.

Adding the interaction energy $\rho_{\rm int}$ to $\Delta\rho_\gamma$, we obtain the total change in the energy density due to the transverse and longitudinal components,

$$\rho_{\rm new} = \Delta\rho_\gamma + \rho_{\rm int}\,. \quad (12)$$

We calculate $\rho_{\rm new}$ and $\rho_{\rm int}$ for an electron-proton plasma at $T = 10^5$ K, $T = 10^4$ K, and $T = 10^5$ K for densities ranging from 10^3–10^{19} cm^{-3}. The densities were chosen so as to assure that the plasma parameter, $g = 1/n\lambda_D^3 < 1$, in order that kinetic theory is valid. For these plasmas, the de Broglie wavelength is less than the distance of closest approach of thermal electrons, which justifies our classical treatment.

In Fig. 1, we plot Δ as a function of the density $10^3 \le n \le 10^{19}$ cm^{-3} for the temperatures $T = 10^3$, 10^4, and 10^5 K. We extended each plot until the density for which $g = 0.3$ was reached. For each of the temperatures, the value of g increases with the density. In

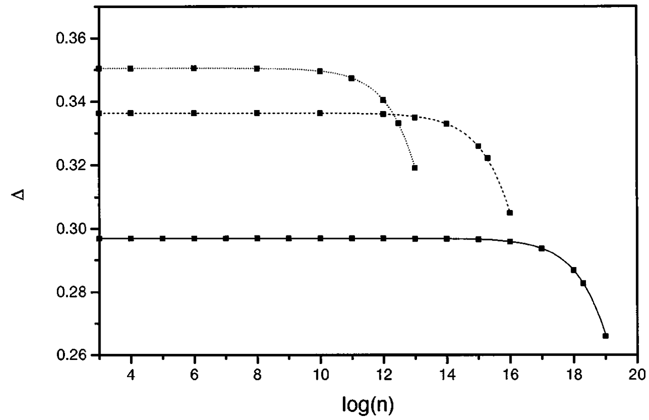

FIG. 1. The correction Δ as a function of density and temperature. The filled curve is for $T = 10^5$ K, the dashed curve for $T = 10^4$ K, and the dotted curve for $T = 10^3$. The curves are evaluated from the analytic expression and the filled squares are the calculated values from Eqs. (1)–(4).

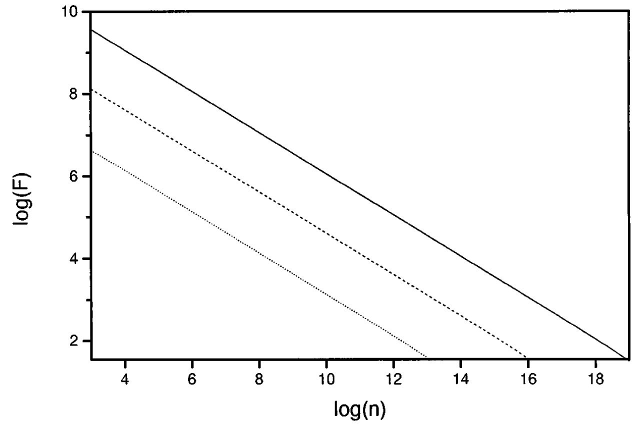

FIG. 2. The deviation of the correction, Δ from the usual one, Δ_0: $F = |\Delta| - |\Delta_0|/|\Delta_0|$. The filled curve is for $T = 10^5$ K, the dashed curve for $T = 10^4$ K, and the dotted curve for $T = 10^3$ K.

the case of $T = 10^5$ K, for example, for $n = 10^3$ cm^{-3}, $g = 9.62 \times 10^{-9}$ and for $n = 10^9$ cm^{-3}, $g = 3.04 \times 10^{-6}$. When $g = 0.3$, $n = 10^{19}$ cm^{-3}. In the case of $T = 10^3$ K, for $n = 10^3$ cm^{-3}, $g = 3.04 \times 10^{-6}$ and for $n = 10^{10}$ cm^{-3}, $g = 9.62 \times 10^{-3}$. When $g = 0.3$, $n = 10^{13}$ cm^{-3}.

We found a very good fit for the results of Fig. 1, using a Fermi-Dirac functional form for the density dependence of Δ, $\Delta(T) = A1/\{\exp[(x/A2) - A3] + 1\}$, with $x = \log(n)$ and $A1 = a_{10} + a_{11}T + a_{12}T^2$; $A2 = a_{20} + a_{21}T + a_{22}T^2$ and $A3 = a_{30} + a_{31}T + a_{32}T^2$. From Fig. 1, we obtain $A1 = 0.3522 - 0.1698(T/10^5) + 0.1145(T/10^5)^2$, $A2 = 0.8255 + 0.4797(T/10^5) - 0.4532(T/10^5)^2$, and $A3 = 17.650 + 33.027(T/10^5) - 26.201(T/10^5)^2$. The curves (filled, dashed, and dotted) are evaluated from the analytic expression; the filled squares are the calculated values of Δ from Eqs. (1)–(4). The fit can be seen to be excellent. In Fig. 2, we plot $F = (\Delta - \Delta_0)/\Delta_0$ as a function of the density, for the temperatures $T = 10^3$, 10^4, and 10^5 K, which shows how Δ differs from the usual correction Δ_0.

The values of Δ that we obtained are positive and larger in absolute value than Δ_0, whereas Δ_0 is negative. This indicates that the energy in the fluctuations dominates the interaction energy of the particles. We observe that F can reach values of a thousand or greater.

The additional transverse energy $\Delta\rho_\gamma$ is completely negligible for these temperatures and densities. For example, for $T = 10^5$ K and $n = 10^{19}$ cm^{-3}, $\Delta\rho \cong 10^{-3}\rho_\gamma$. For this temperature and density, $\rho_{\rm par} = 273\rho_\gamma$ and $\rho_{\rm new}$ is completely dominated by $\rho_{\rm int} = \Delta\rho_{\rm par}$. For example, for $T = 10^5$ K and $n = 10^{17}$ cm^{-3}, $\Delta\rho_\gamma \cong 10^{-7}\rho_\gamma$.

As a check, we calculated $\rho_{\rm int}$, integrating in frequency only up to $\omega = \omega_p$, the plasma frequency ($\ll T$), and integrating in wave number up to $k \le k_D$. As expected, we then found that Δ is equal to Δ_0, the value obtained in previous analysis.

(III) Conclusions and discussion.—We calculated $\rho_{\rm new}$ and $\rho_{\rm int}$ for an electron-proton plasma as a function of density for $T = 10^3$–10^5 K. For many interesting plasmas, we found that $\Delta \gg \Delta_0$. We used the BGK collision term as a rough guide to the inclusion of collisions. The BGK is a model collision term for the Boltzmann collision term. Collisions, however, change the results very little. For example, for $T = 10^5$ K and $n = 10^{10}$ cm^{-3}, the difference in Δ, with or without collisions, is less than 10^{-6}. Since there is no significant difference between the energy density, with or without the collision term, the use of a more accurate collision term than the BGK collision term is not necessary.

Appreciably different values than the usual ones are obtained, for the interaction energy of a plasma, by not assuming $\omega \ll T$. To the first order in g, we found that the energy of an ideal gas needs to be corrected by a positive value, approximately $0.3\rho_{\rm par} = 0.3(3/2)nT$. This results in very different values from the usual ones $\sim(10^{-3}$–$10^{-4})\,(3/2)nT$.

We obtained a general expression for the correction Δ as a function of density and temperature: $\Delta(T) = A1/\{\exp[(x/A2) - A3] + 1\}$ with $x = \log(n)$, $A1 = 0.3522 - 0.1698(T/10^5) + 0.1145(T/10^5)^2$, $A2 = 0.8255 + 0.4797(T/10^5) - 0.4532(T/10^5)^2$, and $A3 = 17.650 + 33.027(T/10^5) - 26.201(T/10^5)^2$.

The total correction to the energy is completely dominated by the interaction energy. For these temperatures and densities, the transverse additional energy is negligible.

Our results may be applied to the plasma before the recombination era, when the plasma had a temperature $T > 10^3$ K and a density $n > 10^3$ cm^{-3}. Since the expansion rate of the Universe (the Hubble parameter) is proportional to the square root of the plasma energy density, our results indicate that the Universe before the recombination era was expanding appreciably faster than previously thought.

The purpose of this work was to demonstrate that $\omega \ll T$ is an extremely strong assumption. By not making this assumption, there is a large change in the energy of the plasma.

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