# Universal $1/f$ Noise from Dissipative Self-Organized Criticality Models

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We introduce a model able to reproduce the main features of $1/f$ noise: hyperuniversality (the power-law exponents are independent on the dimension of the system; we show here results in $d = 1, 2$) and apparent lack of a low-frequency cutoff in the power spectrum. Essential ingredients of this model are an activation-deactivation process and dissipation. [S0031-9007(98)08193-9]

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The voltage drop $V$ on a resistor of resistance $R$ through which a current $I$ is flowing obeys the well-known Ohmic law $V = RI$. Yet, when we look carefully, we discover that such a voltage is not perfectly constant through time. Indeed there are noise fluctuations around $V$. The spectral density of these fluctuations clearly shows a $1/f$ behavior on many decades in the frequency domain. This is a well-known example of $1/f$ noise, one of the most common and widespread features in nature. It appears in a variety of systems ranging from the light of quasars [1] to water flows in rivers [2], music and speech [3], and the already mentioned electrical measurements [4,5]. Despite its ubiquity and universality, a clear and simple explanation for such a behavior is still lacking. Indeed, it is possible to find in the literature some *ad hoc* formulas and theories, but most of them are based on unverified assumptions, or they catch a glimpse of the physics only of some particular system, therefore missing to address the widespread occurrence of the phenomenon [5].

In the search for a universal mechanism of $1/f$ noise, Bak, Tang, and Wiesenfeld (BTW) proposed the new concept of self-organized critical (SOC) systems [6]: These are systems driven by their own dynamics to a state characterized by power-law time and space correlations, and therefore also by power-law ($1/f^{\alpha}$) power spectra. Yet, a number of features of SOC systems do not show agreement with the features of $1/f$ noise: the exponent $\alpha$ is seldom close to 1, and it depends strongly on the dimensionality of the system (at least below the upper critical dimension, which is in general high [7–9]); moreover, in SOC systems power-law time correlations are always found in the presence of power-law (long-range) space correlations, for which there is no evidence in most systems exhibiting $1/f$ noise [10].

In this Letter, we propose a simple model, inspired by a SOC model originally introduced by one of us [11], able to implement some of the current and most accepted ideas on $1/f$ noise and to show a clear $1/f$ behavior independent on the dimension of the system (therefore "hyperuniversal").

The basic model is a continuous version of the BTW sandpile. Given a lattice, to every site $i$ is associated a continuous variable $x_i$ (representing, say, energy). The basic time step of the dynamics consists in changing the value of an energy $x_i$ of a positive random quantity $\epsilon$ [taken from some probability distribution $P(\epsilon)$],

$$x_i(t + 1) = x_i(t) + \epsilon(t), \qquad (1)$$

with the $\epsilon(t)$ variables uncorrelated in time. Whenever this addition step makes an energy $x_i$ greater than a certain value $x_c$, then the quantity $x_i$ is redistributed to the $2d$ nearest neighbors of site $i$,

$$x_j(t, \tau + 1) = x_j(t, \tau) + \frac{x_i(t, \tau)}{2d}, \qquad (2)$$

and the energy $x_i$ is reset to 0. The time variable $\tau$ is used to describe the redistribution process, which is considered to be much faster than the process of addition (1). It is possible that this redistribution drives some other energies to exceed $x_c$, triggering new redistributions. This process (an *avalanche* in the jargon) goes on as long as there are no more energies greater than $x_c$. Then a new quantity is added, as in (1), and time $t$ is increased by 1. It is important to remind the presence of two different time scales: a slow one, corresponding to the addition of energy, and a fast one, corresponding to the redistribution of energies which are above $x_c$. The statistical properties of this model after a transient time are very interesting. The distribution of the energies on the lattice clearly shows a *quantization,* with peaks at about 0, $x_c/2d$, $2x_c/2d, \ldots, (2d - 1)x_c/2d$. Moreover, the distribution of avalanches with respect to their duration (measured on the internal time variable $\tau$) obeys a power law. In general it is possible to show that there are long-range (power-law) space and time correlations. Indeed, this is a SOC model. A further quantity that is interesting to look at is the total lattice energy content $X(t) = \sum_i x_i(t)$. It represents a signal whose power spectrum also obeys some power law in the frequency domain.

This model (as well as the original BTW sandpile) is mainly an activation/deactivation process, which is believed to be one of the main features relevant for the description of $1/f$ noise [12]. Yet, as pointed out above, the exponents depend on the dimensionality of the lattice and are never close to 1 [11]. Therefore the model, as it stands, is not a good candidate to describe $1/f$ noise.

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New ingredients need to be added to the model: the energy is added only on one side of the lattice, defining implicitly a preferred propagation direction for the energy; the second ingredient is dissipation. During redistribution, we added some dissipation in the form,

$$x_j(t,\tau + 1) = x_j(t,\tau) + \frac{x_i(t,\tau)}{2d}(1 - a). \qquad (3)$$

With this new rule avalanches cannot establish anymore long-range correlations throughout the system and are not anymore power-law distributed. In a word, dissipation destroys the self-organized criticality of the system. Yet, some features, such as the *quantization* of the energy levels, survive.

In our implementation of the model, we inject energy on one side of the lattice according to (1), and let it propagate through the lattice following (3). We compute the power spectrum of $X(t)$, finding a clear $1/f$ behavior both in 1 and $2d$ (see Fig. 1) for at least three decades. This is a signature of the desired (and observed in nature) hyperuniversality.

Of course the details of the implementation are relevant up to some level: We take the added random energy $\epsilon$ from a uniform distribution in $[0, \epsilon_{\max}]$, with $\epsilon_{\max} \gg x_c$. Indeed, if $\epsilon_{\max} \leq x_c$ then, on the average, it takes some steps before the energy goes above threshold. Since $\langle \epsilon(t)\epsilon(t')\rangle = \delta(t - t')$, this implies that short time fluctuations of $X(t)$ are uncorrelated and the resulting

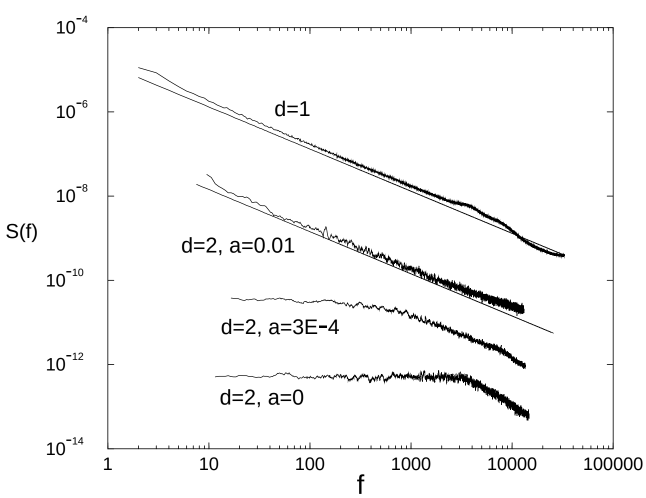


FIG. 1. Log-log plot of the power spectrum of $X(t)$ in $d = 1$ and $d = 2$. The straight lines are $1/f$ power laws drawn for reference. The system size used for the $d = 1$ simulations is $L = 100$, with $\epsilon_{\max} = 10$ and $a = 0.03$; $d = 2$ simulations are performed with $L = 100$, $\epsilon_{\max} = 1$ (energy is added on all the sites $x_{0,i}$, for a maximum possible energy injection of 100) and $a = 0.01$. For comparison, we also add the $d = 2$ power spectrum with a much smaller dissipation ($a = 0.0003$, but with the same size and energy injection regime as before) that shows crossover between clean $1/f$ behavior, and the behavior in the absence of dissipation (lowest curve) that clearly shows no sign of $1/f$ behavior.

power spectrum has a flat tail for high frequencies. Moreover, we find that for larger systems the frequency range where $1/f$ behavior emerges is broader; yet, due to the noncriticality of the model, the transient time to go to stationarity grows fast with the system sizes, forbidding us to explore systems larger than $128 \times 128$ lattice sites in two dimensions. Also, dissipation cannot be too small: indeed, when $a$ becomes very small, then we approach the SOC system, which is characterized by different exponents, and crossover effects emerge. In Fig. 1 we show the $d = 2$ power spectrum with different dissipation regimes. In the absence of dissipation, the power spectrum is essentially flat, with only a small region of power-law behavior $1/f^{1.5}$. A very small dissipation ($a = 0.0003$) clearly gives an intermediate behavior, with a high frequency $1/f$ region, and a low-frequency flat one. Of course, the larger the dissipation, the larger the energy injection must be in order to activate all the lattice.

We believe nonetheless that the way crossover emerges is model dependent. Indeed we believe that the relevant features of the model are nonlinearity (in this case, activation/deactivation of sites) and dissipation.

Additionally, we investigate the power spectrum $S(f,x)$ of the energy at site $x$. We find that, for large $x$, $S(f,x)$ has a scaling form,

$$S(f,x) = e^{\delta x} h(f e^{\delta x}). \qquad (4)$$

From this scaling form we can infer that there is a characteristic time $T(x) \sim e^{\delta x}$ associated with a site at distance $x$ from the origin of the lattice; we can build an intuitive picture of this characteristic time thinking that in order for the energy to propagate from site $x$ to site $x + 1$, it has to overcome some barrier, with a characteristic time to overcome it taken as an Arrhenius law $e^{\delta}$. Then, in order to propagate from the origin down to site $x$, the characteristic time becomes, roughly, of the order of $e^{\delta x}$. As an alternative explanation, due to dissipation, energy has a probability to propagate to a depth $x$ which is exponentially decreasing with $x$, hence an exponential characteristic time associated with $x$.

Because of dissipation, there are no long-range correlations in the system. Therefore, as a first approximation, the energies in different sites are uncorrelated. The total power spectrum can therefore be written as

$$\begin{aligned} S(f) = \sum_x S(f,x) &\sim \int_0^L e^{\delta x} h(f e^{\delta x})\, dx \\ &= \frac{1}{\delta f} \int_0^{f e^{\delta L}} dy\, h(y) \end{aligned} \qquad (5)$$

(indeed the power spectrum of uncorrelated signals is just the sum of the power spectra of the signals, a signature of linear superposition).

We see therefore that the $1/f$ behavior of the power spectrum emerges as the superposition of local power spectra that have nothing to do with $1/f$ noise. The lower cutoff frequency $f_c \sim e^{-\delta L}$ vanishes extremely fast in the thermodynamic limit, accounting for the observed experimental absence of a lower cutoff (whose presence is necessary to have a finite power associated with the signal). Such a superposition mechanism to obtain a $1/f$ power spectrum is strongly reminiscent of the McWhorter model [5,13]: $S(f) = \int S(f, f_c) P(f_c)\, df_c$ with $P(f_c) \sim 1/f_c$ the distribution of the frequencies $f_c$ and $S(f, f_c) \sim f_c/(f^2 + f_c^2)$. In our model we have $df_c/f_c = dx$, accounting for the correct distribution of characteristic frequencies, but we have no explicit form for $S(f, f_c)$ (although also in our case the large $f$ behavior is $1/f^2$; see Fig. 2). The relation $f_c = T_c^{-1} = e^{-\delta x}$ can have many different underlying physical origins, as diverse as tunneling between different traps, jumps between metastable states distributed in space, or dissipation in an activation/deactivation process (as in the present realization). Such a variety of mechanisms giving rise to the good scaling functions (and many others may be conceived) strongly points to the observed widespread occurrence of $1/f$ noise in nature.

Actually, dissipation associated with SOC was already considered in [14,15] and more recently in [16]. There nontrivial power laws in the power spectrum were found, strongly dependent on the dissipation coefficient. Moreover, dissipation and driving were chosen in such a way that the SOC behavior was not destroyed. Actually, many experiments show that there is indeed universality, that the power spectrum is close to $1/f$, and that the long-range time and space correlations typical of SOC systems are absent [10]. All these ingredients, on the contrary, are present in our model, where changing the parameter values gives rise to crossover effects but not to nonuniversality.

Our model shows, moreover, the relevance of a propagation direction (feeding energy from one side of the system and extracting it from the other), and the $1/f$ power spectrum turns out to be the superposition of power spectra that are far from $1/f$.

In conclusion, we have introduced dissipation in a well-known SOC model: As a consequence, the critical behavior of the system is destroyed (but not its self-organization properties, such as the quantization of the energy levels). The resulting power spectrum has a clean $1/f$ behavior both in one and two spatial dimensions: We believe that our one and two dimensional results hint toward the independence of such behavior on the actual dimensionality of the system and to the desired hyperuniversality. We also unveiled that the origin of such a behavior has to be found in the superposition of power spectra with characteristic frequencies $f_c$ suitably distributed in space. Such a distribution is not an input in the model, but emerges due to the directedness properties of our model. The desired distribution of characteristic times typical of the McWhorter model emerges spontaneously in our model. One can conceive many other situations where the distribution of the characteristic frequencies is instead given, and the system only has to cope with it. As a consequence, the present model not only provides a simple and hyperuniversal explanation of $1/f$ noise, but is also suggestive of a wide variety of microscopic physical mechanisms able to give $1/f$ noise.

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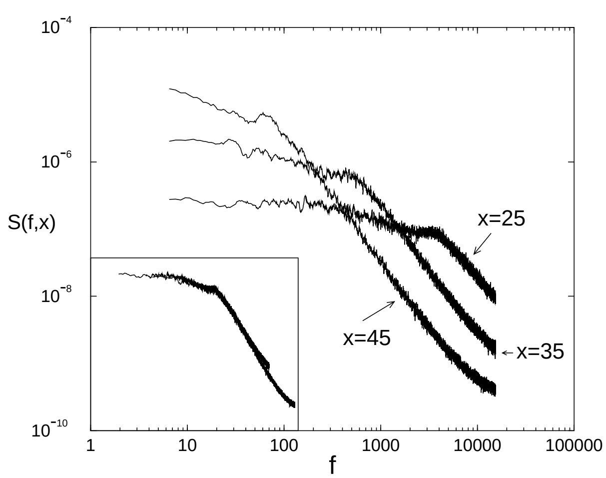


FIG. 2. Log-log plot of the power spectra $S(f, x)$ for $x = 25, 35$, and 45 on a $d = 2$ lattice of $L = 100$, $\epsilon_{\max} = 10$, and $a = 0.01$. The large $f$ behavior has a characteristic slope of $1/f^2$. The collapse in the inset is obtained plotting $e^{-\delta x} S(f, x)$ vs $f e^{\delta x}$ with $\delta = 1.9$. Indeed the universality of the scaling function (4) emerges.

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