

Exact Exponents for the Spin Quantum Hall Transition

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We consider the spin quantum Hall transition which may occur in disordered superconductors with unbroken SU(2) spin-rotation symmetry but broken time-reversal symmetry. Using supersymmetry, we map a model for this transition onto the two-dimensional percolation problem. The anisotropic limit is an $sl(2|1)$ supersymmetric spin chain. The mapping gives exact values for critical exponents associated with disorder averages of several observables in good agreement with recent numerical results. [S0031-9007(99)09270-4]

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Noninteracting electrons with disorder, and the ensuing metal-insulator transitions, have been studied for several decades, and are usually divided into just three classes by symmetry considerations. Recently, the ideas have been extended to quasiparticles in disordered superconductors, for which the particle number is not conserved at the mean field level. Several more symmetry classes have been found [1]. Of particular interest [2–5] is the case in which time-reversal symmetry is broken but global SU(2) spin-rotation symmetry is not (class C of Ref. [1]), and spin transport can be studied. In two dimensions (2D) it can occur in d -wave superconductors. Within this class, a delocalization transition is possible in which the quantized Hall conductivity *for spin* (the linear response coefficient between the spin current and the perpendicular gradient of a Zeeman field) changes by two units, resembling the usual quantum Hall (QH) transition but in a different universality class. When a Zeeman term is introduced which breaks the SU(2) symmetry down to U(1), the transition splits into two that are each in the usual QH universality class.

In this paper, we present *exact* results for a recent model [4,5] for the spin QH transition in a system of noninteracting quasiparticles in 2D. We use a supersymmetry (SUSY) representation of such models, considered previously [6], to obtain a mapping onto the 2D classical bond percolation transition, from which we obtain three independent critical exponents, and universal ratios, exactly. An anisotropic version of the model is also mapped onto an antiferromagnetic $sl(2|1)$ SUSY [7] quantum spin chain. The results are in very good agreement with recent numerical simulations [4,5].

We study the spin QH transition in an alternative description that is obtained from the superconductor after a particle-hole transformation on the down-spin particles [2], which interchanges the roles of particle number and z component of spin, and so particle number is conserved rather than spin. This makes it possible to use a single-particle description, at the cost of obscuring the SU(2)

symmetry. The single-particle energy (E) spectrum has a particle-hole symmetry [1] under which $E \rightarrow -E$, so, when states are filled up to $E = 0$, the positive-energy particle and hole excitations become doublets of the global SU(2) symmetry. In this picture, a (nonrandom) Zeeman term H for the quasiparticles maps onto a simple shift in the Fermi energy to $E = H$ [5], splitting the degeneracy.

The model [4,5] is a network (generalizing Ref. [8]), in which a particle of either spin and with $E = 0$, represented by a doublet of complex fluxes, can propagate in one direction along each link (Fig. 1). The propagation on each link is described by a random SU(2) scattering matrix (the black dot), with a uniform distribution over the SU(2) group; the absence of an additional random U(1) phase here is crucial and implies that the global SU(2) spin-rotation (or particle-hole) symmetry is unbroken. As in Ref. [8], there are two sublattices, A and B , on which the nodes are related by a 90° rotation. Scattering of the fluxes at the nodes (black squares) is described by orthogonal matrices diagonal in spin indices: $S_S = S_{S\uparrow} \oplus S_{S\downarrow}$,

$$S_{S\sigma} = \begin{pmatrix} (1 - t_{S\sigma}^2)^{1/2} & t_{S\sigma} \\ -t_{S\sigma} & (1 - t_{S\sigma}^2)^{1/2} \end{pmatrix}, \quad (1)$$

where $S = A, B$ labels the sublattice, and $\sigma = \uparrow, \downarrow$ labels the spin direction. The network is spatially isotropic when the scattering amplitudes on the two sublattices are related by $t_{A\sigma}^2 + t_{B\sigma}^2 = 1$.

The network has a multicritical point at $t_{A\sigma} = t_{B\sigma} = 2^{-1/2}$ (for the isotropic case). Taking $t_{A\sigma} \neq t_{B\sigma}$ (but

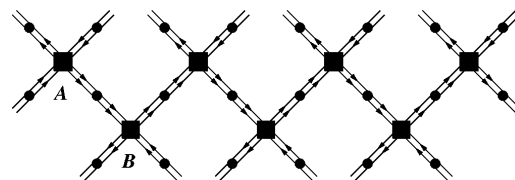


FIG. 1. The network model.

keeping $t_{S\uparrow} = t_{S\downarrow}$ drives the system through a QH transition between an insulator and a QH state, and the Hall conductance (now for charge) jumps from 0 to 2. Making $t_{S\uparrow} \neq t_{S\downarrow}$ breaks the global $SU(2)$ symmetry, and splits the transition into two ordinary QH transitions [4] each in the unitary class. As we will argue later, this perturbation is different from the uniform Zeeman term.

We briefly describe, for the present case, the main steps of the SUSY method for the network models [6]. Transport and other properties of the network, such as its conductance, may be expressed in terms of sums over paths on the network. Such a sum may be written in second-quantized language as a correlation function, $\langle \dots \rangle \equiv \text{STr}(\mathcal{T} \dots U)$, where the supertrace (STr) contains an evolution operator U of an associated quantum 1D problem, \mathcal{T} is a time-ordering symbol, and \dots stands for operators that represent the ends of paths and correspond physically to density, current, etc. In this form, the average can be taken to obtain moments of physical quantities, and we leave this implicit in later notation. In this 1D problem, vertical rows of links of the network become sites, and the vertical direction becomes (imaginary) time (we assume for the present periodic boundary conditions in both directions). The evolution operator U , composed of transfer matrices for links and nodes, acts in a tensor product of Fock spaces of bosons and fermions on each site. The presence of a fermion or boson on a link—i.e., on a site at an instant of discrete time—represents an element of a path traversing that link [6]. Both bosons and fermions are needed to ensure the cancellation of contributions from closed loops. Usually one needs two types of bosons and fermions, retarded and advanced, to be able to obtain two-particle properties. However, the particle-hole symmetry relates retarded and advanced Green's functions [3]. Hence, for the study of *mean* values of simple observables, we need only one fermion and one boson per spin direction per site. [To study fluctuations and other observables, N types of fermion and boson are needed, and the SUSY below becomes $\text{osp}(2N | 2N)$ [1].] We denote them by f_σ, b_σ for the sites related to the links going up (up sites), and $\bar{f}_\sigma, \bar{b}_\sigma$ for the down sites. On the up sites, f_σ, b_σ are canonical, but to ensure the cancellation of closed loops we must either take the fermions on the down sites to satisfy $\{\bar{f}_\sigma, \bar{f}_{\sigma'}^\dagger\} = -\delta_{\sigma\sigma'}$, or similarly for the bosons.

To begin, we consider the spin-rotation invariant case with $t_{S\uparrow} = t_{S\downarrow} = t_S$. In this case, for any realization of the disorder in the scattering matrices, the transfer matrices commute with the sum over sites of the eight generators (superspin operators) of the superalgebra $\text{sl}(2 | 1) \cong \text{osp}(2 | 2)$, similarly to Ref. [6]. The generators for each site are obtained as all bilinears in the fermions and bosons and their adjoints, which are singlets under the random $SU(2)$. These are denoted by [7] $B, Q_3, Q_\pm, V_\pm, W_\pm$, and have similar expressions for the two types of sites. Cancellation of closed loops would require only invariance

under the $\text{gl}(1 | 1)$ subalgebra generated by $B, Q_3, V_-,$ and W_+ . The larger SUSY that exists when $t_{S\uparrow} = t_{S\downarrow}$ is a manifestation of the global $SU(2)$ symmetry.

The transfer matrix describing the evolution on a link, after averaging over the random $SU(2)$ matrices, projects the states on the corresponding site to a three-dimensional subspace of singlets of the random $SU(2)$ [3]. On the up sites, these form the fundamental representation $\mathbf{3}$ of $\text{sl}(2 | 1)$, and we denote them as $|m\rangle, m = 0, 1, 2$. Similarly, on the down sites, the three singlet states form the representation $\bar{\mathbf{3}}$, dual to $\mathbf{3}$, and we call them $|\bar{m}\rangle; m$ is the number of fermions on a site of either type. We find that $|\bar{1}\rangle$ has negative squared norm, $\langle \bar{1} | \bar{1} \rangle = -1$, while the others are positive. Thus, after averaging, we have a horizontal chain of sites with alternating dual representations on the two sublattices and a discrete-time evolution along the vertical direction given by the transfer matrices at the nodes, which will be specified below.

We now consider in detail the node transfer matrix T_S on a single node on sublattice S . After the averaging, it acts in the tensor product $\mathbf{3} \otimes \bar{\mathbf{3}}$ for the two sites. Because of the $\text{sl}(2 | 1)$ SUSY, we find that it takes the form

$$T_S = t_S^2 P_1 + (1 - t_S^2) I \otimes \bar{I}. \tag{2}$$

Here, the first term contains the projection operator $P_1 = |s\rangle\langle s|$ onto the normalized singlet state $|s\rangle = \sum_m |m\rangle \otimes |\bar{m}\rangle$, while in the second term I, \bar{I} are the identity operators on the two sites (note that $\bar{I} = |\bar{0}\rangle\langle\bar{0}| - |\bar{1}\rangle\langle\bar{1}| + |\bar{2}\rangle\langle\bar{2}|$). The two terms in T_S represent the two ways to $\text{sl}(2 | 1)$ -invariantly couple the incoming and outgoing states at the node, such that the incoming state (in the fundamental representation $\mathbf{3}$) flows out unchanged, turning either to the right or the left. They can be represented graphically as shown at the top in Fig. 2.

When we multiply the transfer matrices together and take the supertrace in the tensor product of all sites to calculate the partition function $Z = \text{STr}U$, the result is given by the sum of all contributions of closed loops that

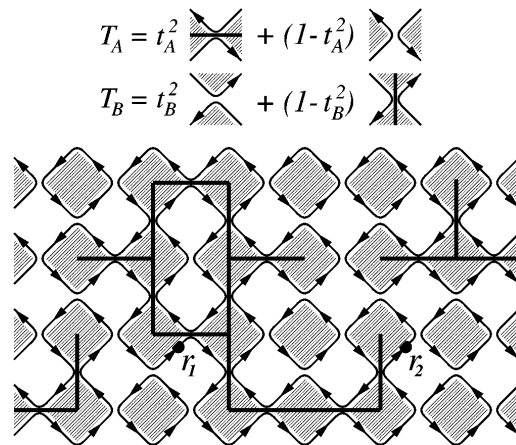


FIG. 2. Graphical representation of the transfer matrices T_S , percolating clusters, and two points r_1, r_2 , on a hull.

fill the links of the network, weighted by factors of either t_S^2 or $(1 - t_S^2)$ for each node. Each loop contributes a factor coming from the sum over the three states that can propagate around the loop, the supertrace $\text{str}1 = 1$ taken in the fundamental **3**. It is also clear that Z is equal to 1, as it is also before averaging.

The sum over loops on the links of the network is equivalent to the bond percolation problem on the square lattice, as follows. In Fig. 2, we shade one-half of the plaquettes of the network in checkerboard fashion. The two terms in T_S possible at each node either do or do not connect the shaded plaquettes, as indicated by the thick undirected line segments. At each A (respectively, B) node we have a horizontal (vertical) line with probability $p_A = t_A^2$ ($p_B = 1 - t_B^2$). Then on the square lattice formed by the shaded plaquettes we have the classical bond percolation problem, and the loops are the boundaries (or ‘‘hulls’’) of the percolation clusters. This SUSY representation of percolation easily generalizes to $\text{sl}(n + 1 | n)$ SUSY, $n \geq 1$, using the $(2n + 1)$ -dimensional fundamental representation and its dual.

Many critical exponents for 2D percolation are known exactly. First, there is the correlation length exponent, which immediately gives the localization length for the spin QH transition,

$$\xi \sim |p_S - p_{Sc}|^{-\nu}, \quad (3)$$

with $\nu = 4/3$ [9]; the critical values are $p_{Ac} = p_{Bc} = 1/2$ in the isotropic case. Then, because the basic operators of our system are the superspins, which act on the states that live on the hulls of the percolation clusters, we should consider the exponents associated with these hulls. These include an infinite set of scaling dimensions for the so-called n -hull operators [10],

$$x_n = (4n^2 - 1)/12. \quad (4)$$

The exponent x_n describes the spatial decay at criticality $\sim |r_1 - r_2|^{-2x_n}$ of the probability that n distinct hulls each pass close to each of the two points r_1 and r_2 . There is also a set of analogous exponents for the same correlators near a boundary [11],

$$\tilde{x}_n = n(2n - 1)/3. \quad (5)$$

We will now relate further physical quantities within our model to percolation exponents, through the SUSY mapping. We write the superspins as a single eight-component object J for either up or down sites. These can be inserted in any links of the network, to obtain a correlator such as $\langle Q_3(r_1)Q_3(r_2) \rangle$, where r_1 and r_2 represent links of the network. Then, using the same graphical expansion, we obtain a sum over loop configurations, now with the positions of the insertions marked on the loops, and, for loops with insertions, the factor 1 is replaced by a supertrace (in the fundamental) of the product of matrices that represent the J 's inserted. We then require only the total probabilities that loops pass through the marked points in various ways. The simplest example is a two-

point function of J 's, which is nonvanishing only if the J 's are on the same loop, because $\text{str}J = 0$ for all components of J . The leading term in the probability that the two points are on the same loop (hull) is governed by the leading one-hull operator in the continuum theory [10], giving

$$\langle (-1)^{i_1+i_2} J(r_1)J(r_2) \rangle \sim |r_1 - r_2|^{-2x_1}, \quad (6)$$

at the transition, where $x_1 = 1/4$ as specified above. The reason for the staggering factors $(-1)^{i_1}$, where i_1 is the site corresponding to r_1 , will become clear momentarily.

It is useful to consider the anisotropic limit of the model. This is defined by $t_A, t_B \rightarrow 0$ with a fixed ratio $t_A/t_B = \epsilon$. Then the transfer matrices T_S may be expanded in t_S and recombined in the exponential. The evolution operator has the form $U \approx \exp(-2t_A t_B \int d\tau \mathcal{H}_{1D})$, where the effective Hamiltonian \mathcal{H}_{1D} describes a 1D superspin chain, with alternating **3** and $\bar{\mathbf{3}}$ representations, and continuous imaginary time τ :

$$\mathcal{H}_{1D} = \sum_i (\epsilon J_{2i-1} \cdot J_{2i} + \epsilon^{-1} J_{2i} \cdot J_{2i+1}). \quad (7)$$

Here, $J \cdot J$ denotes the $\text{sl}(2 | 1)$ invariant product [7]. The transition point, where \mathcal{H}_{1D} for an infinitely long chain is gapless, is now at $\epsilon = 1$. The two-site version of \mathcal{H}_{1D} appeared in Ref. [3].

The sum $\sum_i J_i$ is the generator of global SUSY transformations, and so J_i , viewed as a function of i , is the superspin density on the lattice, which gives a subleading contribution $\sim r^{-2}$ to the J - J correlation at criticality. The one-hull operator must therefore be the staggered part, $(-1)^i J_i$.

The one-hull operators represented by $(-1)^i J_i$ have several physical applications. Components such as $Q_+ = f_{\uparrow}^{\dagger} f_{\downarrow}^{\dagger}$ on the up sites create fermions, so produce ends for the quasiparticle paths. The sum of all such paths between r_1 and r_2 represents the quasiparticle Green's function, G . To obtain nonzero results on averaging, we must multiply the retarded and advanced Green's functions before averaging, but this can be replaced by a spin-singlet combination of our fermions or bosons [3]. The staggered part of this averaged correlator represents the average zero-frequency density-density (‘‘diffusion’’) propagator $\overline{|G|^2}$ (and also the average conductance between two point contacts [12]), which therefore falls as $|r_1 - r_2|^{-1/2}$ at the transition. Moreover, the local density of states $\rho(r, E)$ is represented by another component of the one-hull operator, because both it and the density operator contain wave functions squared, $\sim |\psi|^2$, in the original problem and so scale in the same way. The energy E itself (set to zero hitherto) has scaling dimension $y_1 = 2 - x_1$ because an imaginary part η of E induces a staggered ‘‘magnetic field’’ term $\sum_i \eta \rho_i(i\eta)$ in \mathcal{H}_{1D} . Hence, for the average we have at criticality

$$\overline{\rho(r, E)} \sim |E|^{x_1/y_1} = |E|^{1/7}. \quad (8)$$

Also, since a uniform Zeeman term H causes a shift in the Fermi energy, it induces a correlation length $\xi_H \sim |H|^{-\nu_1}$, where $\nu_1 = 1/y_1 = 4/7$.

We have already identified the value $\nu = 4/3$ of the localization length exponent ν with that in percolation. In terms of \mathcal{H}_{1D} , the effect of a small deviation $\delta \equiv \epsilon - 1$ is to add the perturbation $\delta \sum_i (-1)^i J_i \cdot J_{i+1}$ to the critical \mathcal{H}_{1D} . This term contains the dimer operator $D_i = (-1)^i J_i \cdot J_{i+1}$, which is odd under reflection through any lattice site (parity). The scaling dimension x_2 of the two-hull operator is the same as that of this ‘‘thermal’’ perturbation for the transition; that is, $\nu = \nu_2 = 1/y_2$, $y_2 = 2 - x_2$ [10]. We therefore expect that the two-hull operator is part of a multiplet of staggered two-superspin operators, that are similar to D_i , but are not all $sl(2|1)$ singlets.

As a final perturbation of the critical Hamiltonian, we consider the effect of $t_{S\uparrow} \neq t_{S\downarrow}$. This breaks the global $SU(2)$ symmetry, and breaks the SUSY to $gl(1|1)$. Taking $t_{A\sigma} = t_{B\sigma}$, we find that the effect is to add to \mathcal{H}_{1D} a term $(t_\uparrow - t_\downarrow)^2 \sum_i \hat{J}_i \cdot \hat{J}_{i+1}$, where \hat{J}_i is the four-component set of generators of $gl(1|1)$, and the product is invariant under this algebra. This term is an anisotropy in superspin space. The two QH transitions it produces [4] cannot be seen in our formulation without explicitly introducing both retarded and advanced fermions and bosons, and we will see only exponentially decaying correlations. The correlation length ξ_Δ induced by $\Delta = t_\uparrow - t_\downarrow$ scales as $\xi_\Delta \sim |\Delta|^{-\mu}$, in the notation of Ref. [4], for small Δ . If the spin anisotropy $\hat{J}_i \cdot \hat{J}_{i+1}$ has dimension x' , then we will have $\mu = 2/(2 - x')$. The operator does not appear to be the one-hull operator, and has the opposite parity to the two-hull. However, the operator product of two one-hull operators has the correct parity and might contain this operator. In conformal field theory, the one-hull operator can be represented by $\phi_{2,2}$ in the Kac classification of $c = 0$ Virasoro representations. The fusion rules for this primary field with itself contain the leading nontrivial operator $\phi_{1,3}$, which we view as a subleading one-hull operator, with scaling dimension $\hat{x}_1 = 2h_{1,3} = 2/3$. We suggest that $x' = \hat{x}_1 = 2/3$, which yields $\mu = 3/2$. We further suggest that this operator describes a *random* Zeeman term (with zero mean).

Finally, we note that the average two-probe conductance of our system with open ends [6], and with $t_{S\uparrow} = t_{S\downarrow}$, can be related to the number n of hulls that connect one end to the other (and back). Each such configuration of loops contributes n to the conductance, times 2 for spin, so the mean conductance has the scaling form

$$\bar{g} = 2 \sum_{n=1}^{\infty} n P(n, L/W, L/\xi),$$

where $P(n, L/W, L/\xi)$ is the probability that exactly n hulls run from one end to the other and back, for a system of size L by W . This can be considered both

for periodic and reflecting transverse boundary conditions. At the transition, $\xi = \infty$, and for large L/W , it is known [13] that $P(n, L/W, 0) \sim e^{-2\pi x_n L/W}$ for periodic, and $\sim e^{-\pi \bar{x}_n L/W}$ for reflecting, boundaries. The sum for \bar{g} is dominated by the $n = 1$ term in this limit, so it has the form $\bar{g} \sim e^{-L/\xi_{1D}}$, giving the behavior of the localization length ξ_{1D} , the only parameter that enters in the complete distribution of conductance in this limit [14]. As $L/W \rightarrow 0$, we expect that $\bar{g} \propto W/L$, implying that there is a nonzero critical conductivity.

We may now compare our results with those of recent numerical work. In Ref. [4], the results obtained were $\nu \simeq 1.12$ and $\mu \simeq 1.45$. These are in fair agreement with our predictions, especially for μ where our theoretical argument is less well established. The authors of Ref. [5] study the SUSY spin chain numerically, and find critical exponents $x_1 = 0.26 \pm 0.02$ and $x_2 = 1.24 \pm 0.01$, in excellent agreement with our predictions.

To conclude, we have used SUSY methods to find a remarkable equivalence of a quasiparticle localization problem, the spin quantum Hall transition, to 2D percolation, resulting in the exact values of three exponents, and the universal ratios for the localization length in the 1D limit.

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- [1] A. Altland and M.R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997); M.R. Zirnbauer, J. Math. Phys. (N.Y.) **37**, 4986 (1996).
- [2] T. Senthil *et al.*, Phys. Rev. Lett. **81**, 4704 (1998).
- [3] T. Senthil and M.P.A. Fisher, cond-mat/9810238.
- [4] V. Kagalovsky *et al.*, Phys. Rev. Lett. **82**, 3516 (1999).
- [5] T. Senthil, J.B. Marston, and M.P.A. Fisher, Phys. Rev. B (to be published).
- [6] N. Read (unpublished); I.A. Gruzberg, N. Read, and S. Sachdev, Phys. Rev. B **55**, 10593 (1997).
- [7] M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. (N.Y.) **18**, 155 (1977).
- [8] J.T. Chalker and P.D. Coddington, J. Phys. C **21**, 2665 (1988).
- [9] M. den Nijs, J. Phys. A **12**, 1857 (1979).
- [10] H. Saleur and B. Duplantier, Phys. Rev. Lett. **58**, 2325 (1987).
- [11] H. Saleur and M. Bauer, Nucl. Phys. **B320**, 591 (1989).
- [12] M. Janssen, M. Metzler, and M.R. Zirnbauer, cond-mat/9810319.
- [13] J. Cardy, J. Phys. A **31**, L105 (1998).
- [14] C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).