# Turbulence in Mode-Locked Lasers

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We show that the well-known instability in actively mode-locked lasers with detuning between the resonator round-trip time and the modulator period exhibits a transition to turbulence analogous to fluid flow. We derive a universal normalized detuning of the laser that plays the same role as Reynolds number in fluid flow. This is the first time that the recently proposed theory for the onset of turbulence in hydrodynamics is verified in a system outside of hydrodynamics. [S0031-9007(99)09268-6]

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The transition from laminar to turbulent flow in hydrodynamics has puzzled physicists for more than a hundred years [1]. During the last 5 to 10 years a scenario for the transition to turbulence has been put forward by Trefethen and others [2]. This model does not only give a quantitative description of the kind of instability that leads to a transition from laminar, i.e., highly ordered dynamics, to turbulent flow, i.e., chaotic motion, but it also gives an intuitive physical picture of why turbulence is occurring. According to this theory, turbulence is due to strong non-normal transient growth of deviations from a stable stationary point of the system together with a nonlinear feedback mechanism. The nonlinear feedback mechanism couples part of the amplified perturbation back into the initial perturbation. Therefore, the perturbation experiences the strong growth repeatedly. Once the non-normal transient growth is large enough, a slight perturbation from the stable stationary point renders the system dynamics turbulent. Small perturbations are always present in real systems in the form of system intrinsic noise or environmental noise and in computer simulations due to the finite precision. The predictions of the linearized stability analysis become meaningless in this case. The model case studied here also gives insight into the drifting pulse dynamics in complex Ginzburg-Landau equations [3], which were discovered in convection experiments with binary liquids [4]. The analysis laid out in this Letter might also lead to an improved understanding of noise-sustained convective structures in nonlinear optics [5].

In this Letter, we show for the first time that the scenario for a transition to turbulence, or chaotic motion in general, as presented in [2], is not bound to hydrodynamics but occurs in other systems as well. In particular, we show that the detuned actively mode-locked laser is an excellent example for such a system which in addition can be studied analytically. The dynamics of actively mode-locked lasers is a rather old topic and has been studied in great detail theoretically as well as experimentally [6,7]. However, the detuned case has been studied only either experimentally [8,9] or by numerical simulations [10]. A theoretical approach was never presented. The reason for this lack in analytical approaches to this problem seems to be precisely due to the kind of instability that arises in the detuned system. This type of instability cannot be detected by a linear stability analysis which is widely used in laser theories to prove stable pulse formation. The case studied here might be not only of fundamental interest but might give also an analytical insight into the stability problems associated with asynchronously mode-locked lasers and soliton storage rings which will be important for future high-speed optical communication systems [11].

The equation of motion for the pulse envelope in an actively mode-locked laser with detuning can be written as [7]

$$T_M \frac{\partial A(T,t)}{\partial T} = \Big[ g(T) - l + D_f \frac{\partial^2}{\partial t^2} - M[1 - \cos(\omega_M t)] + T_d \frac{\partial}{\partial t} \Big] A(T,t) . \tag{1}$$

Here, $A(T,t)$ is the slowly varying field envelope whose shape is studied on two time scales. There is the time $T$ which is coarse grained on the time scale of the resonator round-trip time $T_R$ and the time $t$, which resolves the resulting pulse shape. The saturated gain is denoted by $g$, the curvature of the intracavity losses in the frequency domain, which limit the bandwidth of the laser, is given by $D_f$. $M$ is the depth of the loss modulation introduced by the modulator with angular frequency $\omega_M = 2\pi/T_M$, where $T_M$ is the modulator period. The detuning between resonator round-trip time and the modulator period is $T_d = T_M - T_R$. The saturated gain $g$ obeys a separate ordinary differential equation

$$\frac{\partial g(T)}{\partial T} = -\frac{g(T) - g^0}{\tau_L} - g \frac{W(T)}{P_L} . \tag{2}$$

Here, $g^0$ is the small signal gain due to the pumping, $P_L$ the saturation power of the gain medium, $\tau_L$ the gain relaxation time, and $W(T) = \int |A(T,t)|^2 \, dt$ the total field energy stored in the cavity at time $T$.

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The pulses, we expect, have a pulse width much shorter than the round-trip time in the cavity and will be placed in time at the position where the modulator introduces low loss. Furthermore, we restrict our considerations to the case where the modulation depth, $M$, of the modulator is large, such that only during the time of low intracavity loss, which is much shorter than the round-trip time, radiation can build up. In that case, we can approximate the cosine by a parabola and obtain the simplified evolution equation

$$T_M \frac{\partial A}{\partial T} = \left[ g - l + D_f \frac{\partial^2}{\partial t^2} - M_s t^2 + T_d \frac{\partial}{\partial t} \right] A . \quad (3)$$

Here, $M_s = M\omega_M^2/2$ denotes the curvature of the loss modulation at the point of minimum loss and, therefore, it characterizes the modulator strength. The time $t$ is now allowed to range from $-\infty$ to $+\infty$, since the modulator losses make sure that only during the physically allowed range $-T_R/2 \ll t \ll T_R/2$ radiation can build up. Consequently, the domain of the partial differential operators appearing on the right side of Eq. (3) is the space of absolute square integrable complex functions on the real axis.

In the case of vanishing detuning, i.e., $T_d = 0$, the differential operator on the right side of (3), which we denote as the Liouville operator $\hat{L}$, corresponds to the Schrödinger operator of the harmonic oscillator. Therefore, it is useful to introduce the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2}} \left( \tau_a \frac{\partial}{\partial t} + \frac{t}{\tau_a} \right), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( -\tau_a \frac{\partial}{\partial t} + \frac{t}{\tau_a} \right), \quad (4)$$

with $\tau_a = \sqrt[4]{D_f/M_s}$. The Liouville operator $\hat{L}$ is given by

$$\hat{L} = g - l - 2\sqrt{D_f M_s} \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (5)$$

and the evolution equation (3) can be written as

$$T_M \frac{\partial A}{\partial T} = \hat{L} A . \quad (6)$$

Consequently, the eigensolutions of this Liouville operator are the Hermite Gaussians

$$A_n(T,t) = u_n(t) e^{\lambda_n T/T_M} , \quad (7)$$

$$u_n(t) = \sqrt{\frac{W_n}{2^n \sqrt{\pi} n! \, \tau_a}} H_n(t/\tau_a) e^{-(t^2/2\tau_a^2)} , \quad (8)$$

where $\tau_a$ is the pulse width of the Gaussian. The eigenmodes are orthogonal to each other because the Liouville operator is Hermitian in this case.

The round-trip gain of the eigenmode $u_n(t)$ is given by its eigenvalue (or in general by the real part of the eigenvalue) which is given by $\lambda_n = g_n - l - 2\sqrt{D_f M_s}(n + 0.5)$ where $g_n = g^0(1 + \frac{W_n}{P_L T_R})^{-1}$, with $W_n = \int |u_n(t)|^2 \, dt$. The eigenvalues prove that for a given pulse energy the mode with $n = 0$, which we call the ground mode, experiences the largest gain. Consequently, the ground mode will saturate the gain to a value so that $\lambda_0 = 0$ in steady state and all other modes experience net loss, $\lambda_n < 0$ for $n > 0$. This is a stable situation, as can be shown rigorously by a linearized stability analysis [7]. Thus active mode locking with perfect synchronization produces Gaussian pulses with a $1/e$ half-width of the intensity profile given by $\tau_a$. This has been well known since the early work of Siegman [6].

In the case of nonzero detuning $T_d$, the situation becomes more complex. The Liouville operator (5) changes to

$$\hat{L}_D = g - l - 2\sqrt{D_f M_s} \left[ \left( \hat{a}^\dagger - \Delta \right)(\hat{a} + \Delta) + \left( \frac{1}{2} + \Delta^2 \right) \right], \quad (9)$$

with the normalized detuning

$$\Delta = \frac{1}{2\sqrt{2 D_g M_s}} \frac{T_d}{\tau_a} . \quad (10)$$

Introducing the shifted creation and annihilation operators, $\hat{b}^\dagger = \hat{a}^\dagger + \Delta$ and $\hat{b} = \hat{a} + \Delta$, respectively, we obtain

$$\hat{L}_D = \Delta g - 2\sqrt{D_f M_s} \, (\hat{b}^\dagger \hat{b} - 2\Delta \hat{b}) , \quad (11)$$

with the excess gain

$$\Delta g = g - l - 2\sqrt{D_f M_s} \left( \frac{1}{2} + \Delta^2 \right) \quad (12)$$

due to the detuning. Note, the resulting Liouville operator is no longer Hermitian and even not normal, i.e., $[A, A^\dagger] \neq 0$, which causes the eigenmodes to become non-normal [12]. Nevertheless, it is an easy exercise to compute the eigenvectors and eigenvalues of the new Liouville operator in terms of the eigenstates of $\hat{b}^\dagger \hat{b}$, $|l\rangle$, which are the Hermite Gaussians centered around $\Delta$. The eigenvectors $|\varphi_n\rangle$ to $\hat{L}_D$ are found by the ansatz

$$|\varphi_n\rangle = \sum_{l=0}^{n} c_l^n |l\rangle, \quad \text{with } c_{l+1}^n = \frac{n - l}{2\Delta\sqrt{l + 1}} c_l^n . \quad (13)$$

The new eigenvalues are $\lambda_n = g_n - l - 2\sqrt{D_f M_s}(\Delta^2 + n + 0.5)$. By inspection, it is again easy to see that the new eigenstates form a complete basis in $L_2(\mathbb{R})$. However, the eigenvectors are no longer orthogonal to each other. The eigensolutions as a function of time are given as a product of a Hermite polynomial and a shifted Gaussian $u_n(t) = \langle t | \varphi_n \rangle \sim H_n(t/\tau_a) \exp[-\frac{(t - \sqrt{2}\Delta\tau_a)^2}{2\tau_a^2}]$. Again, a linearized stability analysis shows that the ground mode, i.e., $|\varphi_0\rangle$, a Gaussian, is a stable stationary solution. Surprisingly, the linearized analysis predicts stability of the

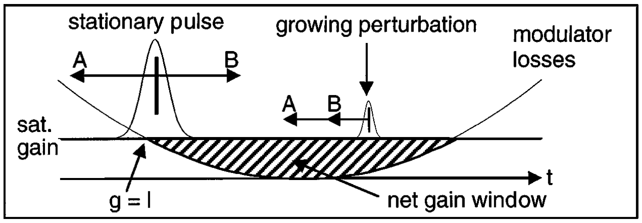


FIG. 1. Drifting pulse dynamics in a detuned actively mode-locked laser for the situation, where the modulator period is larger than the cavity round-trip time. The displacement $A$ is caused by the mismatch between the cavity round-trip time and the modulator period. The displacement $B$ is due to unequal losses experienced by the front and the back of the pulse in the modulator. The gain saturates to a level, where a possible stationary pulse does experience no net gain and loss, which opens up a net gain window following the pulse. Perturbations within that window get amplified while drifting towards the stationary pulse.

ground mode for all values of the detuning in the parabolic modulation and gain approximation. This result is even independent from the dynamics of the gain, i.e., the upper state lifetime of the active medium, as long as there is enough gain to support the pulse. Only the position of the maximum of the ground mode, $\sqrt{2}\,\Delta\tau_a$, depends on the normalized detuning.

Figure 1 summarizes the results obtained so far. In the case of detuning, the center of the stationary Gaussian pulse is shifted away from the position of minimum loss of the modulator. Since the net gain and loss within one round trip in the laser cavity has to be zero for a stationary pulse, there is a long net gain window following the pulse in the case of detuning due to the necessary excess gain. Figure 2 shows a few of the resulting lowest order eigenfunctions for the case of a normalized detuning $\Delta = 0$ in (a) and $\Delta = 0.32$ in (b). These eigenfunctions are not orthogonal as a result of the non-normality of the evolution operator. The non-normality of the operator, $[\hat{L}_D, \hat{L}_D^\dagger] \sim \Delta$, increases with detuning. Figure 3 shows the scalar products between the eigenmodes for different values of the detuning

$$C(m,n) = \left| \frac{\langle \varphi_m | \varphi_n \rangle}{\sqrt{\langle \varphi_m | \varphi_m \rangle \langle \varphi_n | \varphi_n \rangle}} \right| . \qquad (14)$$

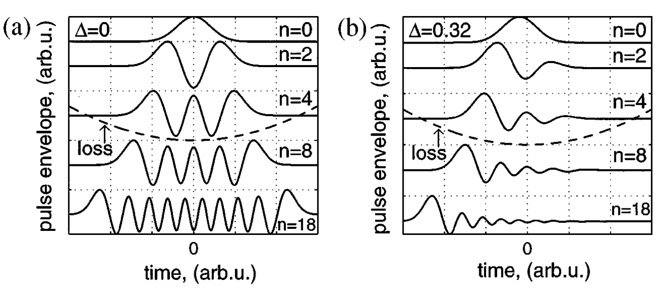


FIG. 2. Lower order eigenmodes of the linearized system for zero detuning, $\Delta = 0$ (a) and for a detuning, $\Delta = 0.32$ (b).

The eigenmodes are orthogonal for zero detuning. The orthogonality vanishes with increased detuning. The recursion relation (13) tells us that the overlap of the new eignemodes with the ground mode is increasing for increasing detuning. This corresponds to the parallelization of the eigenmodes of the linearized problem which leads to large transient gain $\|e^{\hat{L}_D t}\|$ in a non-normal situation [2]. Figure 3(d) shows the transient gain for an initial perturbation from the stationary ground mode calculated by numerical simulations of the linearized system using an expansion of the linearized system in terms of Fock states to the operator $\hat{b}$. A normalized detuning of $\Delta = 3$ already leads to transient gains for perturbations of the order of $10^6$ within 20 000 round-trips which leads to an enormous sensitivity of the system against perturbations. An analytical solution of the linearized system neglecting the gain saturation shows that the transient gain scales with the detuning according to $\exp(2\Delta^2)$. This strong super-exponential growth with increasing detuning determines the dynamics completely. Figure 4 shows the surface of the transition to turbulence in the parameter space of the laser, i.e., critical detuning $\Delta$, the pumping rate $r = g_0/l$, and the ratio between the cavity decay time $T_{\mathrm{cav}} = T_R/l$ and the upper state lifetime $\tau_L$. In this model, we did not include the spontaneous emission. The transition to turbulence always occurs at a normalized detuning of about $\Delta \approx 3.7$ which gives a transient gain $\exp(2\Delta^2) = 10^{12}$. This means that already uncertainties of the numerical integration algorithm are amplified to a perturbation as large as the stationary state itself. Figure 5 shows the Liapunov coefficient [13] of the dynamics as a function of the normalized detuning. It clearly indicates that the dynamics is

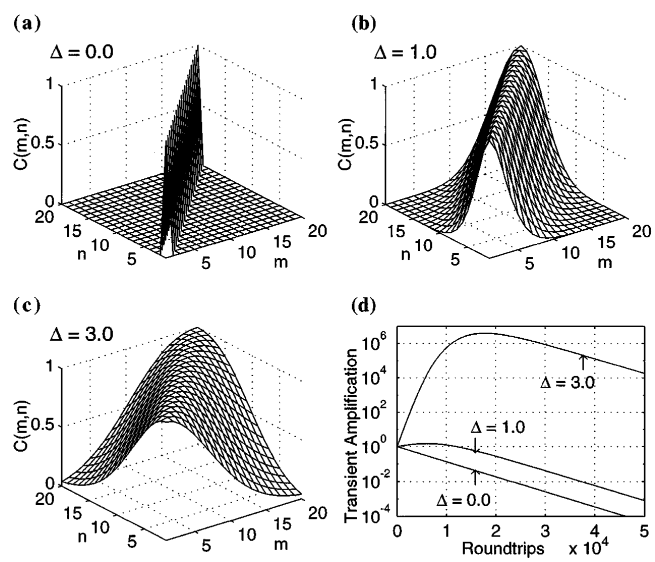


FIG. 3. Scalar products of eigenvectors as a function of the eigenvector index for the cases $\Delta = 0$ (a), $\Delta = 1$ (b), and $\Delta = 3$ (c). (d) shows the transient gain as a function of time for these detunings computed from the linearized system dynamics.

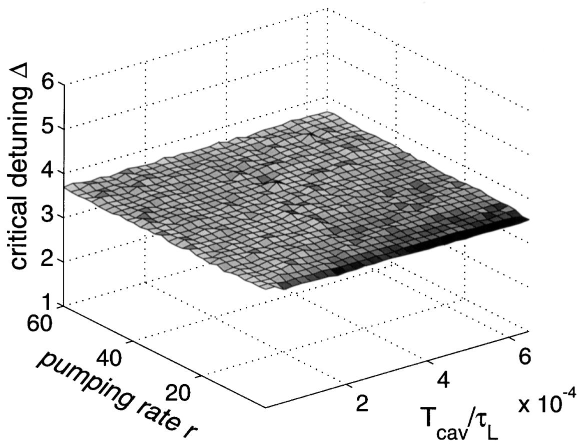


FIG. 4. Critical detuning obtained from numerical simulations as a function of the normalized pumping rate and cavity decay time divided by the upper-state lifetime. The critical detuning is almost independent of all laser parameters shown. The mean critical detuning is $\Delta \approx 3.65$.

chaotic above the critical detuning of about $\Delta_c \approx 3.7$. In the turbulent regime, the system does not reach a steady state, because it is nonperiodically interrupted by a new pulse created out of the net gain window, see Fig. 1, following the pulse for positive detuning. This pulse saturates the gain, and the almost formed steady state pulse is destroyed and finally replaced by a new one. The gain saturation provides the nonlinear feedback mechanism, which strongly perturbs the system again, once a strong perturbation has grown by the transient linear amplification mechanism.

The critical detuning becomes smaller if additional noise sources, such as the spontaneous emission noise of the laser amplifier and technical noise sources, are taken into account. However, due to the superexponential growth, the critical detuning will not depend strongly on the strength of the noise sources. If the spontaneous emission noise is included in the simulation, we obtain the

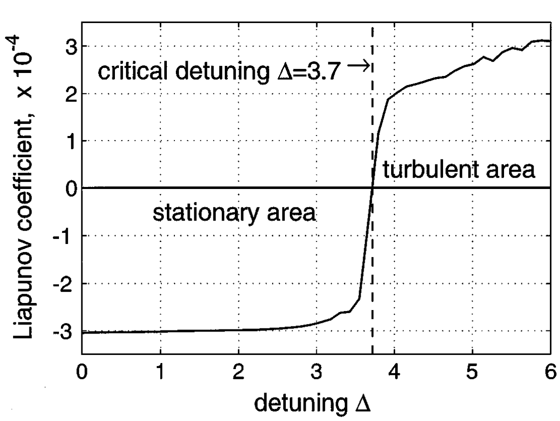


FIG. 5. Liapunov coefficient over normalized detuning.

same shape for the critical detuning as in Fig. 4; however, the critical detuning is lowered to about $\Delta_c \approx 2$.

In conclusion, we have shown that the detuned actively mode-locked laser exhibits a transition to turbulence similar to fluid flow. The dimensionless parameter which governs that transition has been identified as the normalized detuning. The normalized detuning plays the same role for the system investigated, as the Reynolds number does in fluid flow problems. The prerequisites for the transition to turbulence are much easier to grasp for the detuned actively mode-locked laser than in hydrodynamics because of the complexity of the Navier-Stokes equations.

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