Late-Time Decay of Scalar Perturbations Outside Rotating Black Holes

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We present an analytic method for calculating the late-time tails of a linear scalar field outside a Kerr black hole. We give the asymptotic behavior at timelike infinity (for fixed r), at future null-infinity, and along the event horizon (EH). In all three asymptotic regions we find a power-law decay. We show that the power indices describing the decay of the various modes at fixed r differ from the corresponding Schwarzschild values. Also, the scalar field oscillates along the null generators of the EH (with advanced-time frequency proportional to the mode's magnetic number m). [S0031-9007(99)09261-3]

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The *no hair* principle for black holes implies that the gravitational field outside a generic black hole relaxes at late time to the stationary Kerr-Newman geometry. For linear test fields (either scalar, electromagnetic, or gravitational) outside a spherically symmetric Schwarzschild black hole, it was shown by Price [1] that all radiative multipoles die off at late time with a t^{-2l-3} power-law tail [2], where l is the mode's multipole number, and t is the Schwarzschild time coordinate. Later, this result was confirmed using several different techniques, both analytic and numerical [3–6]. The relevance of the perturbative (linear) results to the fully nonlinear late-time behavior was demonstrated in numerical simulations of a fully nonlinear, self-gravitating, spherically symmetric scalar field [7,8].

It is well known, however, that realistic astrophysical black holes are spinning and not spherically symmetric [9]. Therefore, for astrophysical applications it is extremely important to generalize the above analyses from the Schwarzschild background to the more realistic Kerr background. The first progress in this direction has been achieved recently with the numerical simulation of linear fields in the Kerr background, by Krivan *et al.* [10,11]. Yet, a thorough analytic treatment of this problem has not been carried out so far [12,13].

The goal of this Letter is to present an analytic method for calculating the late-time behavior of a linear massless scalar field outside a Kerr black hole. This method was recently applied to the simpler Schwarzschild case as a test bed [6], in which case the well known late-time inverse-power tails were reproduced. In this Letter we outline the application of this method to the Kerr case and present the main results. In particular, we calculate the power indices characterizing the late-time decay of the various modes at future null-infinity, at fixed r, and at the event horizon (EH). Quite interestingly, we find that at fixed r these indices are different than those found in the Schwarzschild case. Full details of the calculations are given in Ref. [14]. Throughout this paper we use the standard Boyer-Lindquist coordinates t, r, θ, φ, and relativistic units $C = G = 1$.

The main difficulty in analyzing perturbations over a Kerr background is the nontrivial dependence on θ. The separation of variables by the Teukolsky equation [15] is applicable only to the Fourier-decomposed field, because the spheroidal harmonics used for the separation of θ and φ explicitly depend on the frequency ω. The final goal is to calculate the late-time decay of the field, along with its angular dependence, in terms of the time t (i.e., in the time domain). At the late-time limit $t \to \infty$, we expect the tails to be dominated by the very small Fourier frequencies, $\omega \to 0$, for which the spheroidal harmonics reduce to the *spherical* harmonics Y_l^m. This motivates us to carry out the entire analysis in terms of the spherical harmonics. The difficulty is, however, that due to the breakdown of spherical symmetry, modes of different l (but the same m) are coupled; namely, there are "interactions" between modes. The main challenge is to handle this interaction and to analyze its effect on the late-time decay.

In principle, it is possible to carry out the analysis in the frequency domain and then Fourier-integrate over all frequencies to recover the late-time behavior in the time domain, as was done in the Schwarzschild case [4,5]. We find it simpler, however, to carry out the entire analysis in the time domain. To overcome the difficulties caused by the interaction between modes, in the first part of the analysis we use an iteration scheme in which we iterate over the interaction term (along with the other curvature-induced terms in the field equation). In the second part of the analysis we use the *late-time expansion,* which is essentially an expansion in inverse powers of advanced time. (Both methods are generalizations of those used in Refs. [6,16] for the spherical case.)

The Klein-Gordon field Φ in Kerr geometry obeys

$$[(r^2 + a^2)^2/\Delta - a^2 \sin^2\theta]\Phi_{,tt} - (\Delta\Phi_{,r})_{,r} + 4Mar\Delta^{-1}\Phi_{,t\varphi} + [a^2/\Delta - (\sin\theta)^{-2}]\Phi_{,\varphi\varphi} - (\sin\theta)^{-1}(\Phi_{,\theta}\sin\theta)_{,\theta} = 0\,, \quad (1)$$

where M and a are, correspondingly, the black hole's mass and specific angular momentum, and $\Delta \equiv r^2 - 2Mr + a^2$. Decomposing Φ into spherical harmonics

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in the standard manner, $\Phi = \sum_{lm} \phi^{lm}(t,r)Y_l^m(\theta,\varphi)$, and defining $\Psi^{lm} \equiv \sqrt{r^2 + a^2}\,\phi^{lm}$, the original field equation (1) becomes (for each m)

$$\Psi^l_{,uv} + V_K^{lm}(r)\Psi^l + i\,\frac{mMar}{(r^2+a^2)^2}\,\Psi^l_{,t} + \frac{a^2\Delta}{(r^2+a^2)^2}\,[c_0\Psi^l_{,tt} + c_-\Psi^{l-2}_{,tt} + c_+\Psi^{l+2}_{,tt}] = 0\,, \qquad (2)$$

where c_0, c_-, and c_+ are certain coefficients depending on l and m (with $c_- = 0$ for $m = l, l-1$; no other coefficients vanish), and where

$$4V_K^{lm} = \Delta(r^2+a^2)^{-4}(2Mr^3 + a^2r^2 - 4Ma^2r + a^4) - (r^2+a^2)^{-2}[m^2a^2 - l(l+1)\Delta]\,. \qquad (3)$$

The coordinates u and v are defined by $u = t - r_*$ and $v = t + r_*$, with $r_*(r)$ obeying $dr_*/dr = (a^2 + r^2)/\Delta$. [In Eq. (2), and also in most of the equations below, we omit the index m for brevity. Note that due to the axial symmetry, modes with different m do not interact.]

Equation (2) is an infinite set of coupled equations for the various modes of the field, with the last two terms in the square brackets describing the interactions between modes of different l.

The setup of initial data for the evolution problem is similar to that used in Ref. [6] for the Schwarzschild case [see Fig. 2 and Eq. (7) therein]. That is, we assume that Φ is specified along a pair of hypersurfaces $v = 0$ and $u = u_0$. For convenience we consider a situation of an outgoing pulse at $v = 0$, which starts immediately after the outgoing ray $u = u_0$. We further assume that $-u_0 \gg M$ and that the pulse is arbitrarily shaped but relatively short, which considerably simplifies the analysis [6,16]. Although this type of initial data is not the most general one, it is nevertheless reasonably generic, and we expect the resultant asymptotic behavior to be characteristic of the generic situation. We also assume here that the initial outgoing pulse has a rather generic angular distribution, so it includes all the spherical harmonics (and especially the component $l = 0$).

To evolve these initial data and analyze the late-time behavior, we shall proceed in two steps. In the first step, we calculate the late time (i.e., $u \gg M$) form of the field at future null-infinity ($v \to \infty$). In the second step we derive an expression for the field at a fixed r at $t \gg M$ (and also along the EH at $v \gg M$).

Asymptotic behavior at future null-infinity.—We now apply the iteration scheme introduced in Refs. [6,16], and decompose Ψ^{lm} as

$$\Psi^{lm} = \sum_{N=0}^{\infty} \Psi_N^{lm}. \qquad (4)$$

The components Ψ_N^{lm} are defined by the hierarchy of equations

$$(\Psi^l_{N,})_{,uv} + V_0^l\Psi_N^l = S_N^l\,, \qquad (5)$$

where $S_0^l \equiv 0$ and (for $N \geq 1$)

$$S_N^l \equiv -(\delta V_K^l)\Psi^l_{N-1} - i\,\frac{mMar}{(r^2+a^2)^2}\,(\Psi^l_{N-1})_{,t} - \frac{a^2\Delta}{(r^2+a^2)^2}\,[c_0\Psi^l_{N-1} + c_+\Psi^{l+2}_{N-1} + c_-\Psi^{l-2}_{N-1}]_{,tt}\,, \qquad (6)$$

along with the initial conditions $\Psi_0^l = \Psi^l$ and $\Psi^l_{N\geq1} = 0$ at $v = 0$ and $u = u_0$. Here, $V_0^l(r_*)$ is the Minkowski-like potential defined in Ref. [6] as a function of r_* [Eqs. (8), (60) therein], and $\delta V_K^l(r) \equiv V_K^l - V_0^l$. Formal summation over N recovers the original field equation and initial data for the complete fields Ψ^l.

The field equation (5), together with the above initial conditions, constitutes a hierarchy of initial-value problems for the various functions Ψ_N^{lm}, which, in principle, we may solve one by one (first for $N = 0$, then for $N = 1$, etc.). Notice that the potential $V_0(r_*)$ (and hence the entire $N = 0$ equation) is independent of the spin parameter a. The solution of this equation, the function Ψ_0^l, is given explicitly in Ref. [6] (see Sec. IV therein). This function decays exponentially at late time, so it does not contribute to the power-law tail. Rather, it serves as a source for terms $\Psi_{N\geq1}$, which do provide power-law tails. For each $N \geq 1$, the field equation can be formally solved in terms of a Green's function:

$$\Psi_N^l(u,v) = \int_{u_0}^{u} du' \int_0^{v} dv'\, G^l(u,v;u',v')S_N^l(u',v')\,, \qquad (7)$$

where $G^l(u,v;u',v')$ is the (retarded) Green's function associated with the zero-order operator $\partial_u\partial_v + V_0^l$. An analytic expression for G was derived in Sec. V of Ref. [6]. This, in principle, enables the solution of the field equation (5) for all N and l.

The functions Ψ_1^l will primarily concern us here, because it is the term $N = 1$ which dominates the late-time behavior of Ψ^l at null-infinity in the generic situation. It is convenient to consider separately the contributions from the various terms in Eq. (6) [through Eq. (7)] to Ψ_1^l. Consider first the contribution from the term proportional to δV_K^l. This potential can be expressed as $\delta V_K^l = \delta V_S^l + \delta V_a^l$, where δV_S^l is the (a-independent) corresponding Schwarzschild contribution, and δV_a^l is an a-dependent correction term. A direct calculation shows that at $r \gg M$, δV_a^l decays faster than δV_S^l by a factor proportional to a^2/Mr_*. An explicit evaluation of the integral in Eq. (7) then yields that this extra factor leads to an extra u^{-1} factor in the late-time asymptotic behavior of Ψ_1^l at null-infinity [14]. Thus, the dominant contribution to Ψ_1^l from δV_K^l at null-infinity, which we denote by $\hat{\Psi}_1^l$, is the same as in the Schwarzschild case [cf. Eq. (58) in Ref. [6]]:

$$\hat{\Psi}_1^l(u \gg M) \cong A_l u^{-l-2}, \qquad (8)$$

where A_l is given explicitly in [6] as a linear functional of the l component of the initial pulse.

Consider next the contribution to Ψ_1^l due to the other part of S_1^l, i.e., the part containing t derivatives in Eq. (6). A direct evaluation of the integrals in Eq. (7) shows that the late-time contribution of this part at null-infinity is proportional to u^{-l-3} or smaller [14] and is hence negligible. The only exception is the contribution from $\Psi_{0,tt}^{l-2}$ (for $l \geq 2$), which is proportional to u^{-l-2}, too, so it does not cause a qualitative change in the asymptotic decay (8). Moreover, the coefficient of this term is reduced by a factor $\propto (a/u_0)^2 \ll 1$ compared to A_l, so the overall tail amplitudes are still given by Eq. (8) at the leading order.

We still need to consider the terms $N \geq 2$. A complete analysis of these terms has not been carried out yet. In the Schwarzschild case, simple considerations suggest that at null-infinity all these terms decay like u^{-l-2}, though with coefficients smaller than that of Ψ_1 by a factor $(M/u_0)^{N-1}$. [This was also verified by numerical simulations [6], which also suggested convergence of the sum (4) at null-infinity.] Hence the $N \geq 2$ terms do not alter the power index, and, moreover, in the case $-u_0 \gg M$ considered here, they do not significantly affect the coefficients. All these considerations apply to the Kerr case as well [14,17]. Assuming that the terms $N \geq 2$ indeed behave in that manner, we find that at late time, the scalar field at null-infinity is given by

$$\Psi^l \cong A_l u^{-l-2} \text{ (null infinity, } u \gg M), \tag{9}$$

where the coefficients A_l coincide with those of Eq. (8) to the leading order in M/u_0.

Derivation of Φ *at* $r =$ const*: the late-time expansion.*—We now derive an expression for the late-time behavior at any fixed r outside the black hole and along the EH, accurate to the leading order in M/t or M/v, respectively. To that end we employ the late-time expansion used in the Schwarzschild case (cf. Ref. [6]):

$$\phi^{lm}(r, v) = \sum_{k=0}^{\infty} F_k^{lm}(r) v^{-k_0-k}. \tag{10}$$

As it turns out, this expansion is consistent with the field equation, with the regularity condition at the EH, and with the form of the field at null-infinity. The parameter $k_0 > 0$ is by definition l independent and will later be determined through matching to null-infinity. [For $l > 0$ some of the first terms in the sum (10) vanish, as will become apparent below.]

Substituting the expansion (10) in the field equation (2) and collecting terms of the same power in v, the partial differential equation becomes an infinite coupled set of *ordinary* equations for the unknown functions $F_k^l(r)$,

$$[\Delta(F_k^l)']' + [a^2m^2/\Delta - l(l+1)]F_k^l = Z_k^l \equiv 2(k_0 + k - 1)[(r^2 + a^2)(F_{k-1}^l)' + (r - 2imMar/\Delta)F_{k-1}^l + 2a^2(k_0 + k - 2)(c_0 F_{k-2}^l + c_+ F_{k-2}^{l+2} + c_- F_{k-2}^{l-2})], \tag{11}$$

where a prime denotes d/dr (and where $F_{k'<0}^l \equiv 0$). Here, too, the source term Z_k^l exhibits interactions with modes $l' \neq l$. However, since Z_k^l depends only on terms $F_{k'}^{l'}$ with $k' < k$, the system (11) is effectively decoupled, as the various ordinary equations may be solved one at a time. It is possible to formally write down the general solution for any F_k^l via the Green's function method [14]. Consider first the term $k = 0$, which dominates the overall late-time asymptotic behavior at fixed r. The function $F_{k=0}^l$ satisfies a homogeneous equation (actually the stationary field equation), whose general solution is given by $F_0^l = a_l P_l^{-\gamma}(\rho) + b_l P_l^{+\gamma}(\rho)$. Here, a_l and b_l are (yet) arbitrary constants, $\rho \equiv (2r - r_+ - r_-)/(r_+ - r_-)$, where $r_\pm \equiv M \pm (M^2 - a^2)^{1/2}$, and $P_l^{\pm\gamma}$ are the two complex-conjugated *associated Legendre functions of the first kind* [18] with an imaginary index $\gamma = im[2a/(r_+ - r_-)]$. The functions $P_l^{\pm\gamma}$ (which are special cases of the Hypergeometric function) have the form

$$P_l^{\pm\gamma}(\rho) = \mathcal{P}_l^{\pm\gamma}(\rho) \times [(\rho + 1)/(\rho - 1)]^{\pm\gamma/2}, \tag{12}$$

in which $\mathcal{P}_l^{\pm\gamma}$ are (complex) polynomials of order l (nonvanishing at $r \to r_+$). For $m \neq 0$ these functions oscillate rapidly towards the EH ($r \to r_+, \rho \to 1$),

$$P_l^{\pm\gamma}(r \to r_+) \propto (\rho - 1)^{\mp\gamma/2} \propto \exp(\mp im\Omega_+ r_*), \tag{13}$$

where $\Omega_+ \equiv a/(2Mr_+)$.

One of the two coefficients a_l, b_l is to be determined from the regularity condition at the EH. Here one must recall that the Boyer-Lindquist coordinate φ is singular at the EH. Using the regularized azimuthal coordinate $\tilde{\varphi}_+ \equiv \varphi - \Omega_+ t$ [19] instead, one finds

$$e^{im\varphi} = [e^{im\tilde{\varphi}_+} e^{im\Omega_+ v}]e^{-im\Omega_+ r_*}. \tag{14}$$

Since the factor in square brackets is regular at the EH (but the next factor is not), it follows from the regularity condition that $b_l = 0$; hence $F_0^l = a_l P_l^{-\gamma}(\rho)$.

The parameter a_l is to be determined through matching to null-infinity. The asymptotic form of F_0^l as $r \to \infty$ is $F_0^l(r \gg M) \propto r^l$. Substitution in Eq. (10) (taking into account the contribution of the terms $k > 0$ as well, which are not negligible at null-infinity, as explained in [6]), one obtains at null-infinity $\Psi^l \propto a_l u^{l+1-k_0}$ [14]. Matching this expression to Eq. (9) for $l = 0$ yields $k_0 = 3$. This value of k_0 yields a consistent matching for any l, implying $a_{l\geq 1} = 0$ (that is, the modes $l \geq 1$ are excited only at $k > 0$). One finds that the dominant mode $l = 0$ decays like $v^{-3} \propto t^{-3}$ at fixed r (and large t), as in the Schwarzschild case.

The interaction between modes has a crucial effect on the decay rate of modes $l \geq 2$. Without this interaction, one can verify that a mode l, m is excited at $k = 2l$, leading to a decay rate t^{-2l-3} (as in the Schwarzschild case). The interaction changes this situation. Consider, for example, the mode $l = 2, m = 0$. This mode has a vanishing source term $Z_{k=1}^{l=2}$ (as $F_{k=0}^{l=2} \equiv 0$), and one can show that $F_{k=1}^{l=2}(r) \equiv 0$. On the other hand, at $k = 2$ there is a nonvanishing source term $Z_{k=2}^{l=2} \propto c_- F_{k=0}^{l=0}$, which necessarily leads to a nonvanishing function $F_{k=2}^{l=2}(r)$. Thus, the decay rate of this mode at fixed r is $v^{-k_0-k} = v^{-5} \propto t^{-5}$, which differs from the corresponding Schwarzschild rate, t^{-7}. This simple consideration is easily extended to all other modes m, l, and one finds [14]

$$\Psi^{lm} \propto t^{-l-|m|-3-q} \ (\text{fixed } r, \ t \gg M, |r_*|), \qquad (15)$$

where $q = 0$ for even $l + m$ and $q = 1$ for odd $l + m$. The late-time behavior of a mode l, m at the EH (expressed in regular coordinates) is found to be

$$\Psi^{lm} Y_l^m(\theta, \varphi) \propto Y_l^m(\theta, \tilde{\varphi}_+) e^{im\Omega_+ v} v^{-l-|m|-3-q}. \qquad (16)$$

[This power index may be changed if the relevant function $F_k^l(r)$ happens to vanish as $r \to r_+$.]

In summary, the late-time behavior of the various modes in the three asymptotic regions is given in Eqs. (9), (15), and (16). Our analysis indicates two phenomena special to the Kerr case:

(A) Oscillations along the EH—cf. Eq. (16).

(B) The interaction between modes: Because of this interaction, the power index at fixed r is $l + |m| + 3 + q$. This result was demonstrated numerically for $l = |m|$ in [10], and recently also for several $l > |m|$ modes in [20]. Note, however, that the significance of the spherical harmonic functions used here is limited: These functions are related to the specific Boyer-Lindquist coordinates r, θ and not to an underlying symmetry group. Yet, the separability of the field equation in these coordinates at the late time limit (which concerns us here) signifies the r, θ variables as natural ones.

Throughout this paper we have assumed that the initial pulse includes all the modes (and, in particular, the dominant mode $l = 0$). In nongeneric situations in which the low-l modes are absent at the initial data, the interaction between modes may dominate the overall late-time behavior already at null-infinity. For example, assume that the initial pulse is a pure $l = 2, m = 0$ mode. Then, without the interactions, at null-infinity Ψ would be dominated by $\Psi_{N=1}^{l=2} \propto u^{-4}$. However, the interaction excites (at $N = 2$) an $l = 0$ mode with a u^{-2} tail, which dominates the late-time behavior. This behavior has been observed numerically by Krivan *et al.* [10]. In Ref. [14] this phenomenon will be discussed in more detail, along with its implications to the asymptotic behavior at fixed r.

We should emphasize that despite the relative simplicity of the calculation scheme presented here, the mathematical question of convergence of the various expansions involved is still open (though there is evidence for convergence). This is the situation even in the Schwarzschild case. An additional subtlety arises from the fact that the characteristic hypersurfaces of the Kerr geometry slightly deviate from those of the iteration scheme used above.

The generalization of the present analysis to electromagnetic and gravitational perturbations will be presented elsewhere [14]. Here we mention only the peculiar behavior of $s > 0$ Newman-Penrose fields along the event horizon. The late-time behavior for $s > 0$ is generically dominated by modes $l = s$. For $0 < |m| \leq s$ these modes decay at the horizon like v^{-2s-3}, whereas the axisymmetric mode $l = s, m = 0$ decays slower, like v^{-2s-4}. This phenomenon is demonstrated and explained in Ref. [21].

Note added.—After this manuscript was submitted, Hod analyzed the mode coupling in Kerr spacetime [22]. His results for a scalar field ($s = 0$) are fairly similar to ours; however, for $s > 0$ Hod's analysis does not recover the phenomenon mentioned above.

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