

## Universality and Crossover of Directed Polymers and Growing Surfaces

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We study Kardar-Parisi-Zhang surfaces on Euclidean lattices and directed polymers on hierarchical lattices subject to different distributions of disorder, showing that universality holds, at odds with recent results on Euclidean lattices. Moreover, we find the presence of a slow (power-law) crossover toward the universal values of the exponents and verify that the exponent governing such crossover is universal too. In the limit of a  $(1 + \epsilon)$ -dimensional system we find, both numerically and analytically, that the crossover exponent is  $1/2$ . [S0031-9007(99)09043-2]

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The problem of directed polymers in random media (DPRM) [1] has attracted much attention in the last ten years, both as a paradigm in the area of disordered systems and for the richness of its connections with other systems, in particular noisy surface growth governed by the Kardar-Parisi-Zhang (KPZ) equation [2,3]. Both problems show space and time scaling behavior, and the connection between the two manifests through a correspondence between their exponents. Within the KPZ context, it has been recently proposed that the model could be nonuniversal [4]: The exponents characterizing the surface growth depend on the details of the driving noise, at least for a substrate dimension  $d \geq 2$ . These results have been obtained via a lattice formulation of the KPZ equation where the strong coupling limit (which is the nontrivial regime of surface growth) is shown to be completely equivalent to the ground state problem of DPRM's. Thus, we can expect nonuniversality in the DPRM context as well. It is therefore interesting to reexamine this issue both within the KPZ context and for DPRM's.

Before proceeding further, it is useful to recall the definition of the exponents within the two contexts, and their relations. Starting from a flat substrate of characteristic linear size  $L$ , the roughness of a KPZ surface grows initially as  $W(t, L) \sim t^\beta$ ; at longer times it saturates, and it scales with  $L$  as  $W(t, L) \sim L^\chi$ . The characteristic time  $\tau$  between the two regimes scales with the size of the system as  $\tau \sim L^z$ . These exponents are not independent: the relation (rooted in the Galilean invariance of the KPZ equation)  $\chi + z = 2$  holds in every dimension [2,3]. Moreover, consistency imposes  $\chi = z\beta$ . Therefore there is just a single independent exponent. In the language of DPRM's, the exponent  $\beta$  governs the fluctuations of the ground state energy,  $\Delta E = \sqrt{\langle (E_{GS} - \langle E_{GS} \rangle)^2 \rangle} \sim t^\omega$  with  $\omega = \beta$  (here we use the Greek alphabet letters commonly used in the literature); the transverse wandering fluctuations of the ground state polymer are governed by the exponent  $\zeta = 1/z$ ,  $\Delta l \sim t^\zeta$ . The relation  $\omega = 2\zeta - 1$  holds in every dimension (it is related to  $\chi + z = 2$ ).

The exponents are known exactly only for  $d = 1$ :  $\beta = 1/3$  and  $z = 3/2$  ( $\omega = 1/3$ ,  $\zeta = 2/3$ ). At present there are no exact solutions for  $d \geq 2$ , and our knowledge

of the exponents relies on numerical simulations. Yet, the situation is not completely clear even numerically: As it has been pointed out recently [4], in the context of surface growth (KPZ equation) different distributions of disorder seem to give different values of the exponents. This result has been interpreted as a case of nonuniversality. Indeed, using a distribution of the energies as

$$p(x) = \frac{(1 - \alpha)}{2} (1 - |x|)^{-\alpha},$$

$$x \in (-1, 1), \quad \alpha > -1, \quad (1)$$

in [4] it was found that the values of the exponents depended on the value of  $\alpha$  in (1). In [4] the exponents were calculated only for  $d \geq 2$ , assuming that universality holds in  $d = 1$ , where the exponents are exactly known (it is worth mentioning that the exact knowledge of the exponents is based on taking a Gaussian distribution of disorder).

To settle this problem, we have performed simulations of surface growth on  $d = 1$  Euclidean lattices. We measure the  $\beta$  exponents ruling the growth in time of the roughness  $W(t)$  starting from a flat substrate (the other exponents can be obtained from the above mentioned relations). We used three different values of  $\alpha$ , namely  $\alpha = 0.5, 0.75, 0.9$  (in each case we also verified that we are still far from saturation). As can be seen from Fig. 1, a power law with exponent  $1/3$  (the theoretical one) is not suited to fit the numerical power laws obtained. A naive fit would give instead  $\beta = 0.28, 0.26$ , and  $0.24$ , respectively. Even fitting an exponent on the last four points with  $\alpha = 0.5$  would give  $\beta = 0.30$  (results for the most commonly used case, the uniform distribution with  $\alpha = 0$ , are not given since the fitted exponent is  $\beta = 0.31$  and the crossover is less evident). To understand whether this is a case of nonuniversality or of crossover, we also analyze the running exponents (see Fig. 2a). If the numerical points shown in Fig. 1 are taken at fixed time ratios  $t_{n+1}/t_n = r$ , then the running exponents are defined as

$$\beta_n = \log_r \frac{W(t_{n+1})}{W(t_n)}, \quad (2)$$

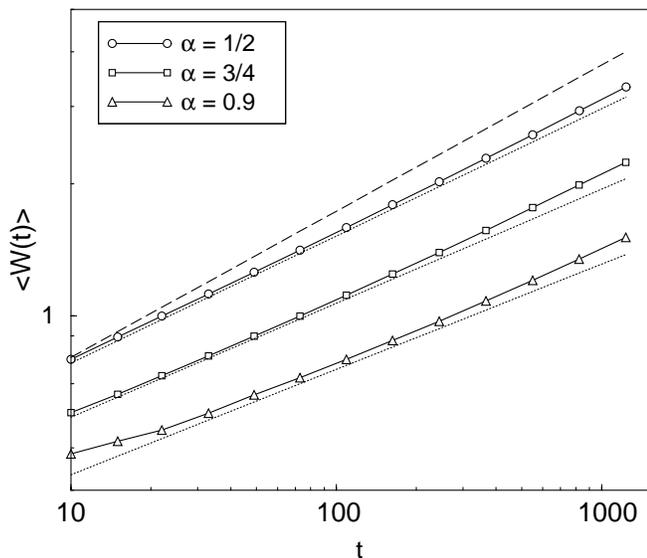


FIG. 1. Roughness of KPZ surfaces for exponents  $\alpha = 0.5, 0.75,$  and  $0.9$  in Eq. (1). The dashed line represents a power law with exponent  $\beta = 1/3$ , the dotted lines represent tentative fits. Each numerical point is the average from 10 000 independent disorder realizations.

where, in the present case,  $r = 3/2$ . We find that the running exponents approach their universal value  $\beta = 1/3$ . In a log-log plot (Fig. 2b) it is then easy to see that such an approach is ruled by a power law with a universal (independent of  $\alpha$ ) exponent  $\gamma = 0.23 \pm 0.02$ .

The results obtained are amenable of different interpretations: either the  $d = 1$  case is nonuniversal too, against naive expectations, or it is indeed affected by extremely slow crossovers. The analysis of the running exponents

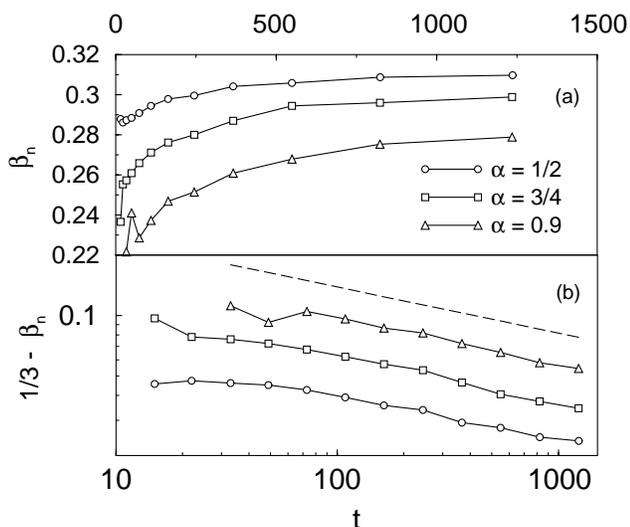


FIG. 2. (a) Running exponents for the roughness as in Fig. 1. (b) Log-log plot of the difference of the running exponents from the theoretical value  $1/3$ . The dashed line represents a power law with exponent  $0.23$ .

seems to sustain the latter hypothesis. The behavior in higher dimensions is unfortunately much more difficult to extract. Indeed, we have also performed simulations in  $d = 2$ , but the results, although similar, are somehow less conclusive due to numerical limitations: in order to get reliable statistics in  $d = 1$ , each numerical point is averaged over 10 000 independent realizations. Moreover, universality in the crossover is evident only for times  $\geq 100$ . Matching both conditions for  $d \geq 2$  goes beyond the capabilities of our present computing facilities.

To investigate the high dimensional behavior of this crossover we resort then to analytical and numerical calculations on hierarchical lattices, where the same behavior emerges.

The main idea of hierarchical lattices is that it is possible to build them iteratively, given a fundamental bond-block transformation, where a bond is substituted by a block as in Fig. 3. The inverse process can be seen as a coarse graining transformation. Indeed, using this transformation, it can be shown that real-space renormalization becomes exact on hierarchical lattices. We exploit this property to write the corresponding renormalization for the ground state of polymers on hierarchical lattices with bond disorder taken from a given distribution.

Given the ground state energy distribution at a certain renormalization step  $n$ , it is possible to compute the distribution of the ground state energy at step  $n + 1$  via the equation

$$P_{n+1}(x) = bQ_n(x) \left[ \int_x^\infty Q_n(x') dx' \right]^{b-1}, \quad (3)$$

where  $b$  is the number of sides as from Fig. 3.  $Q_n(x)$  is the convolution of two probability distributions  $P_n(x)$ ,  $Q_n(x) = P_n(x) * P_n(x)$ . The right-hand side (rhs) of Eq. (3) represents the probability that one of the  $b$  sides has an energy  $x$ , while all the others have an energy greater than  $x$ . Because of the choice of the minimum energy, this is also the probability that the ground state energy is  $x$  at step  $n + 1$ .

The number of sides  $b$  can be related to a fractal dimension of the lattice. Indeed, at every renormalization step we rescale the length of the lattice of a factor 2, and the volume of a factor  $2b$ . Therefore the dimension of the system can be related to  $b$  via the formula  $D = 1 + \log_2 b$  (or, correspondingly,  $d = D - 1 = \log_2 b$ ).

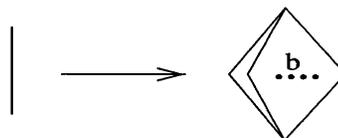


FIG. 3. Iterative procedure to build a hierarchical lattice: the bond on the left schematically represents the lattice at step  $n$ ; then it is used to build the lattice at step  $n + 1$ , represented on the right. In the picture only three of the  $b$  sides are explicitly drawn.

Given Eq. (3), the calculation of the exponent  $\omega$  reduces in principle to a numerical iteration of (3) given a starting distribution  $P_0(x)$ . The ground state energy fluctuation exponent can be computed as

$$\omega_n = \log_2 \frac{\sigma_{n+1}}{\sigma_n}, \quad (4)$$

where  $\sigma_n$  is the variance of  $P_n(x)$ . Asymptotically, the running exponent  $\omega_n$  tends to a constant  $\omega_\infty$ . Possible nonuniversalities should then emerge using different distributions. Yet, calculations of  $\omega_\infty$  show that different distributions give the same asymptotic values up to  $b = 20$  (corresponding to a substrate dimension  $d = 4.32\dots$ ).

As a by-product, we see that hierarchical lattices of fractal dimension  $D = d + 1$  give values of the  $\omega$  exponents close to the results on Euclidean lattices of the same dimension [ $D = 2$ ,  $\omega_{\text{hier}} = 0.30(1)$ ,  $\omega_{\text{eucl}} = 1/3$ ;  $D = 3$ ,  $\omega_{\text{hier}} = 0.22(1)$ ,  $\omega_{\text{eucl}} = 0.24(1)$ ;  $D = 4$ ,  $\omega_{\text{hier}} = 0.15(1)$ ,  $\omega_{\text{eucl}} = 0.16(1)$ ].

Although universality holds for directed polymers on hierarchical lattices, we find that the running exponents  $\omega_n$  show a power-law crossover toward their asymptotic value  $\omega_\infty$  (see Fig. 4a):

$$|\omega_\infty - \omega_n| \sim 2^{-\gamma(n+1)}, \quad (5)$$

$2^{n+1}$  being the length of the lattice.

The power-law exponent  $\gamma$  does not depend on the details of  $P_0(x)$ , that instead are responsible of the amplitude of the crossover. We used Gaussian distributions and distributions of the form (1), as considered in [4].

In Fig. 5 we show the running exponent approach to  $\omega_\infty$  for different values of  $b$  (all the numerics in this case have been done using Gaussian distributions; as already explained above, universality of the crossover exponent is verified). We find that, as  $b \rightarrow 1$  then  $\gamma \rightarrow 1/2$ .

We verify that indeed  $\gamma(b = 1^+) = 1/2$  via a  $b = 1 + \epsilon$  expansion of Eq. (3) on the same lines in Ref. [5]: there a Gaussian microscopic distribution  $P_0(x)$  was used, and no crossover effects were discovered. We use instead gamma distributions of the kind  $P(x; \nu) = \frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x}$ , defined for  $x > 0$ : these distributions are *partially stable* under convolution; indeed,  $P(x; \nu) * P(x; \nu) = P(x; \nu' = 2\nu)$ , that is, the convolution of two gamma distributions is still a gamma distribution (although characterized by a different exponent  $\nu'$ ).

We assume that the distribution at iteration  $n$  is given by

$$P_n(x) = P_n^0(x) [1 + \epsilon \Phi_n(x)], \quad (6)$$

where  $P_n^0(x)$  is the distribution at iteration  $n$  for the one-dimensional system ( $b = 1$ ), simply given by  $P_n^0(x) = P_{n-1}^0(x) * P_{n-1}^0(x) = Q_{n-1}^0(x)$ . The perturbation  $\Phi_n(x)$  must satisfy the relation

$$\int_0^\infty \Phi_n(x) P_n^0(x) dx = 0. \quad (7)$$

We expand Eq. (3) for small  $\epsilon$ , keeping only terms of order  $\epsilon$ :

$$Q_n^0(x) + \epsilon Q_n^0(x) \Phi_{n+1}(x) = Q_n^0(x) + \epsilon \left\{ Q_n^0(x) + Q_n^0(x) \ln \int_x^\infty Q_n^0(x') dx' + 2P_n^0(x) * [P_n^0(x) \Phi_n(x)] \right\}. \quad (8)$$

The linear terms in  $\epsilon$  are a recursion relation for the perturbation  $\Phi(x)$ ,

$$Q_n^0(x) \Phi_{n+1}(x) = Q_n^0(x) + Q_n^0(x) \ln \int_x^\infty Q_n^0(x') dx' + 2P_n^0(x) * [P_n^0(x) \Phi_n(x)]. \quad (9)$$

All the terms in (9) are proportional to  $Q_n^0(x)$  except for the last term on the rhs. We deal with this last term with the *ansatz* equation

$$P_n^0(x) * [P_n^0(x) \phi_{n,s}(x)] = \lambda_{n,s} Q_n^0(x) \phi_{n+1,s}(x), \quad (10)$$

whose purpose is to extract from the left-hand side expression a term proportional to  $Q_n^0(x)$  that can be then simplified in (9). If among the solutions of Eq. (10) it is possible to find sets  $[\phi_{n,s}]$  that are complete and orthonormal for any  $n$ , then we can write  $\Phi_n(x) =$

$\sum_s a_{n,s} \phi_{n,s}(x)$  and (9) becomes a recursion relation for the  $a_n$  coefficients.

Any function of the form

$$\phi_{n,s}(x) = \frac{(-1)^s}{\sqrt{s!}} \sqrt{\frac{\Gamma(\nu_n)}{\Gamma(\nu_n + s)}} \frac{1}{P_n^0(x; \nu_n)} \frac{d^s}{dx^s} \times [x^s P_n^0(x; \nu_n)] \quad (11)$$

satisfies Eq. (10) with “eigenvalues”

$$\lambda_{n,s} = \sqrt{\frac{\Gamma(\nu_n + s) \Gamma(\nu_{n+1})}{\Gamma(\nu_n) \Gamma(2\nu_n + s)}}. \quad (12)$$

Equation (11) is nothing other than Rodrigues’ formula for the generalized Laguerre polynomials, properly normalized, that are known to be orthogonal and complete with weight  $P_n^0(x)$  [6]. The first three such polynomials are

$$\begin{aligned} \phi_{n,0}(x) &= 1, & \phi_{n,1}(x) &= (x - \nu_n) / \sqrt{\nu_n}, \\ \phi_{n,2}(x) &= [x^2 - 2(\nu_n + 1)x + \nu_n(\nu_n + 1)] / \sqrt{2\nu_n(\nu_n + 1)}. \end{aligned} \quad (13)$$

Substituting the series expansion of  $\Phi_n(x)$  in (9), it is possible to read the iteration equations for the coefficients  $a_{n,s}$ :

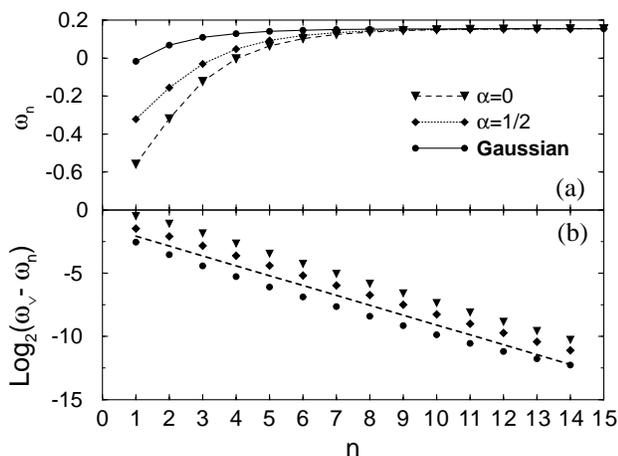


FIG. 4. (a) Running exponents for  $b = 8$  with different disorder distributions, namely Gaussian and as from Eq. (4); (b) Log-log plot of the difference of the same running exponents from their asymptotic value, clearly showing the power-law behavior with  $\gamma = 0.80(2)$ . The value  $b = 8$  has been chosen for numerical stability.

$$\begin{aligned} a_{n+1,0} &= 1 + K_{n,0} + 2\lambda_{n,0}a_{n,0}, \\ a_{n+1,s} &= K_{n,s} + 2\lambda_{n,s}a_{n,s} \quad s > 0, \end{aligned} \quad (14)$$

with

$$K_{n,s} = \int_0^\infty dx P_{n+1}^0(x) \phi_{n+1,s} \ln \int_x^\infty P_{n+1}^0(x') dx'. \quad (15)$$

It is straightforward to show that  $K_0 = -1$ ; moreover, from Eq. (7) we find  $a_{0,0} = 0$  and therefore  $a_{n,0} = 0$  for any  $n$ , as required. It is also possible to show that  $2\lambda_{n,s} < 1$  if  $s > 2$  (indeed the leading behavior for large  $\nu_n$ , that is for large  $n$ , is  $2\lambda_{n,s} \sim 2^{1-s/2}[1 + (s-1/2)s/2\nu_n]^{1/2}$ ). Therefore all the iterations equations (14) with  $s > 2$  converge and are asymptotically irrelevant for the scaling behavior. We are left with  $s = 1, 2$ . Indeed  $a_{n,1}$  and  $a_{n,2}$  are the only coefficients that are relevant for the computation of  $\omega_n$  from (4).

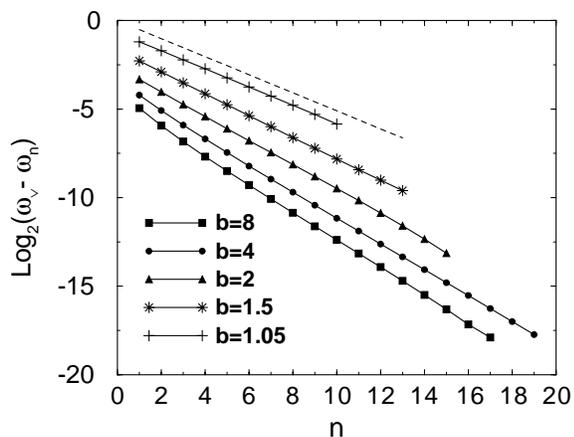


FIG. 5. Log-log plot of the difference of the running exponents from their asymptotic value for different values of  $b$ . The dashed line represents a power law with exponent  $-1/2$ . All the data have been obtained using Gaussian distributions.

We can then compute the variance  $\sigma_n^2$  as

$$\begin{aligned} \sigma_n^2 &= \langle x^2 \rangle_n - \langle x \rangle_n^2 \\ &= \nu_n + \epsilon(\sqrt{2\nu_n(\nu_n + 1)}a_{n,2} + 2\sqrt{\nu_n}a_{n,1}). \end{aligned} \quad (16)$$

We can write then

$$\frac{\sigma_{n+1}^2}{\sigma_n^2} = 2 \left[ 1 + \epsilon \left( \sqrt{2} K_{n,2} + 2 \frac{K_{n,1}}{\sqrt{\nu_{n+1}}} \right) \right], \quad (17)$$

where we keep only the leading and next-to-the-leading terms. From (17) the order  $\epsilon$  correction to the  $\omega$  exponent can be computed. Indeed,  $\sigma_{n+1}^2/\sigma_n^2 = 2^{2\omega_n}$ , and since  $\nu_{n+1} = 2^{n+1}\nu_0$ , we can write

$$\omega_n = \frac{1}{2} + \epsilon \frac{\sqrt{2}}{\ln 2} K_{n,2} + \epsilon \frac{2}{\ln 2 \sqrt{\nu_0}} K_{n,1} 2^{-(n+1)/2}. \quad (18)$$

Furthermore, both  $K_{n,1}$  and  $K_{n,2}$  have a power-law convergence to their asymptotic value, with exponent  $1/2$  (this result can be obtained using a Gaussian approximation in their evaluation). As a result the rhs of (18) converges with exponent  $\gamma = 1/2$  to  $\omega_\infty$ .

In conclusion, we have found numerically that both the KPZ equation on  $d = 1$  Euclidean lattices and DPRM on hierarchical lattices show universal behavior, being the exponent  $\omega$  independent on the details of the disorder distribution. Yet, the presence of nontrivial power law (therefore “slow”) crossover effects has been unveiled. Moreover, we have shown that the crossover exponent  $\gamma$  is universal. We have also shown numerically and analytically that  $\gamma \rightarrow 1/2$  when  $b \rightarrow 1$ .

Since we find the same qualitative slow crossover behavior both on  $d = 1$  Euclidean lattices and on hierarchical lattices of any dimension, it is reasonable to think that the same behavior is present also for  $d \geq 2$  Euclidean lattices, providing a way out of the nonuniversality claimed in [4].

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