Alternating Locking Ratios in Imperfect Phase Synchronization

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In periodically driven chaotic dynamical systems with a broad distribution of intrinsic time scales, perfect phase synchronization cannot be reached. Long segments of evolution during which the phase of a chaotic variable follows the phase of the driving force are interrupted by short segments of phase drift. We demonstrate that this drift is another short-lived synchronized state; its onset is caused by the passage near the long unstable periodic orbits whose frequencies are locked by external force in ratios different from 1:1. [S0031-9007(99)09190-5]

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Synchronized processes are widespread in nature and in technical applications [1]; in recent years it has been understood that coupling can synchronize not only periodic, but also chaotic systems. Depending on the type and strength of the coupling, several stages of chaotic synchronization can be distinguished. The strongest one is the complete synchronization when the states of coupled systems converge, irrespective of the mismatch in initial conditions [2]. In a wider context, the state of the driven system is a function of the state of the driving one ("generalized synchronization" [3]).

Among the forms of chaotic synchronization, phase synchronization is at the closest to the literal meaning of the word: *syn* + *chrónos*= (sharing the) common time. This relatively weak form does not assume that the amplitudes of motions in the interacting subsystems get locked; intrinsic time scales of subsystems, however, become commensurate [4]. In case of a periodically forced chaotic system, phase synchronization adjusts characteristic times of the chaotic motion to the force period; in previously studied examples it was observed as a permanent locking between the force phase and the suitably defined phase of the chaotic motion [5,6]. To enable such dynamics, the range of intrinsic time scales should be narrow. However, in a natural system these time scales can differ strongly; this impedes maintainment of the constant phase locking. Recently, studies on a human cardiorespiratory system provided an example in which locking ratios changed in the course of time [7]. In this Letter, we describe the mechanism which enables the phase of the driven system to switch back and forth between different locking ratios. Analysis by means of unstable periodic orbits shows that the seeming phase jumps are in fact segments of different kinds of synchronized motion, whose alternation is caused by the overlap of Arnold tongues with different winding numbers.

Since in phase space the attracting sets of dissipative dynamical systems are bounded, a typical trajectory repeatedly returns to any selected region of the attractor. It is often helpful to replace a continuous flow by the Poincaré map which the flow induces on an appropriate surface of section; dynamics of this map is complemented by the dependence of return time on the coordinates along the Poincaré surface. We denote the part of the orbit between two returns onto the surface as "orbit turn" and introduce the mean frequency of returns as $\omega \equiv 2\pi \lim_{T\to\infty} N(T)/T$, where $N(T)$ is the number of returns within the time T. The next step leads to phase Φ: each new orbit turn should add 2π to Φ. For the long-time effects, details of the phase evolution within a single turn are of little importance, and interpolation is often sufficient. A finer definition would relate the instantaneous phase to the Hilbert transform of the chaotic variable [4,8], with instantaneous frequency being the time derivative of this phase; as recently checked [5], in situations typical for chaotic synchronization, application of both definitions leads to practically the same results.

In many cases the variation of return times along the Poincaré surface is modest; examples are provided by attractors originating from the period-doubling scenario, like the Rössler attractor. Here, a relatively weak forcing with period close to the mean return time of the autonomous system suppresses the variations and imposes perfect phase synchronization so that the phase of the chaotic motion follows the phase of the force [9,10].

One can expect a different picture in systems with large variations of return times; these are inevitable, e.g., when an attractor of an autonomous system includes a saddle point. The latter situation is known from problems of optics (Raman scattering, lasers with saturable absorbers, optothermal devices [11]), thermal convection in certain configurations [12,13], or reductions of weakly dissipative 1- and 2-dimensional complex Ginzburg-Landau equations [14]. Our example is the archetype of deterministic chaos: the Lorenz equations [15]

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz + E\cos(\Omega t)$$

perturbed by a periodic external force with amplitude E and frequency Ω; here, $\sigma = 10$, $r = 28$, and $b = 8/3$ are the original parameter values from [15].

Without forcing ($E = 0$) a typical trajectory wanders along two symmetric lobes of the attractor: large loops around two saddle-focus points alternate in an irregular

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pattern. Chaotic orbits come arbitrarily close to the local stable manifold of a saddle point at the origin, and the return times on a Poincaré plane can be arbitrarily high.

An indication of phase synchronization would be the vanishing difference between the mean frequency ω and the frequency of the driving force Ω within the range of the values of Ω [4]. Indeed, a certain plateau in the plot of $\omega - \Omega$ vs Ω appears (Fig. 1; here, $E = 6$); however, this plateau is neither horizontal nor lies at zero. (For comparison we show the perfectly horizontal plateau centered around $\Omega = 24.92$, calculated for $r = 210$, $E = 3$. At high values of r the attractor of the autonomous Lorenz equations does not include a saddle point [16], return times are confined within a narrow gap, and dynamics can be easily synchronized already by weak forcing [17]).

Similar approximate plateaus are observed for the other values of the amplitude $E > 2.4$. According to numerical data, here in the chaotic orbit long segments, in which its phase follows the phase of the force, alternate with short intervals where the two phases go apart. Typical evolution of difference between the phases of the system and the force is presented in Fig. 2. The "staircase" built of long "stairs" connected by abrupt "jumps," summarizes 12 300 orbit turns. Each stair corresponds to a phase-synchronized state; departures from horizontal lines caused by crudeness in determination of phase annihilate in the net effect. Transitions between the stairs look like occasional phase slips. At the start the system and the force share the same phase value; within the first stair they remain roughly in-phase; within the second stair the force has a phase lag of 2π, or, in other words, of one period; within the third stair the time lag makes two periods of the force, and so on. In the course of time, not only 2π jumps but also the rarer 4π jumps are observed. Each transition is preceded by a sharp decline downwards; this means that just before it the phase trajectory rotates slowly. In the rough assumption that a transition happens within a single turn, $\varepsilon \equiv |\omega - \Omega|/\Omega$ yields a ratio of the number of nonsynchronized turns N_n to that of the synchronized turns N_s: $\varepsilon \cong N_n/N_s$. Smallness of ε inside the plateau confirms the prevalence of synchronized segments within a chaotic orbit and allows us to view the corresponding states as "nearly" or "imperfectly" phase synchronized.

We explain the mechanism of this imperfection with the help of unstable periodic orbits. Embedded into chaotic sets as a kind of "skeleton," these orbits allow one to resolve fine details of dynamics [18,19], not only in explicitly given maps and flows, but also in noisy experimental data [20] and spatiotemporal patterns [21].

Also the distribution of return times can be characterized in these terms. For the orbit which has the period τ and closes after n returns onto the Poincaré plane (we call n the "orbit length," as in [9]), its individual frequency is defined as $\omega_{\mathrm{i}} = 2\pi n/\tau$.

In the presence of weak forcing, each periodic state can be viewed as an individual periodically forced oscillator: trajectories wind on the invariant 2-torus born from the closed curve of the autonomous system. Since the latter is unstable, the torus in the weakly driven system is also unstable. In the parameter space, phase-locking regions (Arnold tongues) correspond to rational ratios between the driving frequency Ω and the individual frequency ω_{i} of the periodic orbit; often only the main tongue $\Omega = \omega_{\mathrm{i}}$ is relevant for the applications. Inside the tongue, two closed orbits coexist on the surface of the torus: the attracting "phase-stable" one and the repelling one; on the edges of the tongue they disappear via the tangent bifurcation. Outside the tongues the trajectories are dense on the torus; the motion is not synchronized.

If the frequencies ω_{i} for different periodic orbits of the autonomous system are close to each other, the main Arnold tongues of these orbits overlap. If a domain common to all tongues exists, all periodic motions are locked there by the force in the same ratio; the perfect phase synchronization occurs. On their way over the attractor, chaotic orbits repeatedly visit the neighborhoods of tori. There they approach the respective phase-stable solution and move along it until the instability of the torus repels them to another torus. Just outside the overlap domain, synchronized motions are interrupted by phase slips, in a kind of intermittency which has been called "eyelet," since each violation of synchrony owes

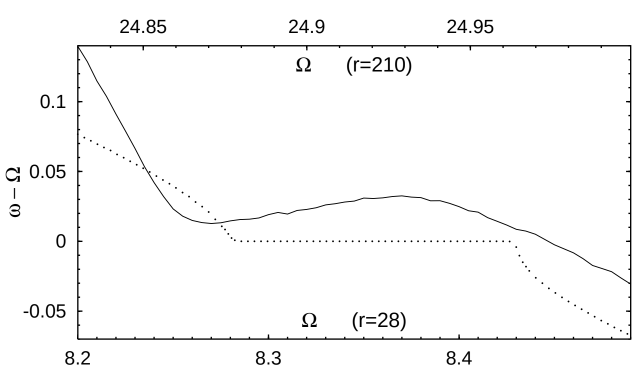

FIG. 1. Perfect and imperfect phase synchronization; solid line: $r = 28$, $E = 6$; dotted line: $r = 210$, $E = 3$.

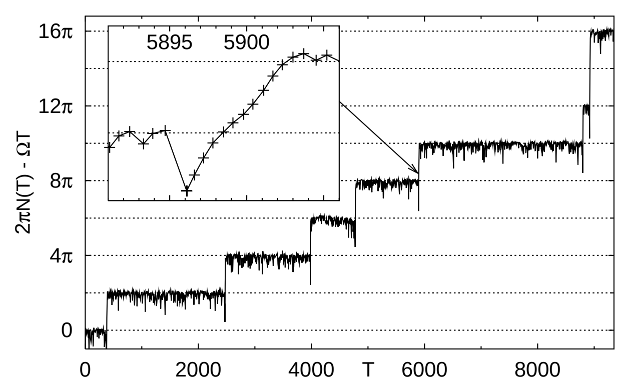

FIG. 2. Time growth of difference between the phase of chaotic motion $2\pi N(T)$ and the phase of driving force ΩT; $\Omega = 8.29$, $E = 6$.

to a precise hitting of the small vicinity of a nonlocked torus [6].

Figure 3 shows the frequency distribution for unstable periodic orbits embedded into the attractor of autonomous Lorenz equations. Among all the orbits of length n, the highest ω_{i} is attained by the orbit which performs one large turn on one of the attractor lobes and $n - 1$ turns on the opposite lobe. During the former, the orbit slows down in the region of the saddle point; from here, it is reinjected relatively close to the saddle-focus point where the rate of rotations in the phase space is especially high. Fast rotations counterbalance the initial slowdown, and the return time averaged over the orbit remains low.

The values of ω_{i} are bounded from above by the imaginary part of the Jacobian eigenvalues at the saddle foci; this bound is unreachable for periodic orbits since the saddle foci themselves lie outside the attractor. The lowest ω_{i} can be arbitrarily small: an infinitesimal shift of a parameter leads through the countable number of homoclinic bifurcations [16], and the periods of closed orbits born from these bifurcations can be arbitrarily large. However, the proximity to bifurcations makes these orbits extremely unstable and hardly relevant: the contribution of a periodic orbit into dynamics is inversely proportional to its positive Lyapunov exponent [19] which diverges at the homoclinicity. As for the relevant periodic orbits of short and moderate length, the band of their frequencies is bounded from below. Nevertheless, the variations among ω_{i} are strong: the distance between the extrema ω_{max} and ω_{min} exceeds 12% of the mean value $\omega = 8.365\dots$ (the latter estimate is an average over 10^7 turns of a chaotic orbit). This broad scattering of intrinsic times of the autonomous system hampers synchronization by external forcing.

Let us magnify a vicinity of a phase "jump" (Fig. 2 inset). Here, crosses denote intersections of the orbit with the Poincaré plane; between them, the phase is interpolated linearly. What seemed to be an instantaneous phase slip proves to be a process which requires not less than a dozen orbit turns. A long interval between two returns onto the Poincaré plane at the very start of the transition indicates a very slow turn. Several very short intervals follow; they not only compensate the phase lag caused by the initial slowdown but transport the system upwards to the next stair. This scenario reminds us of the periodic orbits from the upper branch of the frequency distribution (Fig. 3), with their hoverings near the saddle origin, their fast rotations around the saddle foci, and their high individual frequencies.

This guess is supported by the structure of the Poincaré section of the attractor; an appropriate coordinate is the value of the force phase Ψ at the moment of intersection with the Poincaré plane [9]. Since Ψ is cyclic, one observes, in general, either an unbounded drift along Ψ, or, for the state of perfect phase synchronization, a pattern localized in Ψ [6,9,10]. In our case (Fig. 4 shows several 2π-periods of the pattern), the system spends most of the time in densely populated central stripes; at times, it drifts along the diffuse "whiskers" downwards.

To elucidate this drift, we identify inside the pattern several phase-stable periodic orbits: the squares correspond to one of the orbits of length 7, locked by external force in the ratio 1:1, the circles denote the orbit of length 15, and the crosses show the orbit of length 20. The two latter orbits stem from the upper branch of the distribution (Fig. 3); their ω_{i} in the autonomous case equal 8.978 and 9.028, respectively. Under the values of Ω corresponding to imperfect synchronization, these orbits are locked by the force in the ratio 14:15 and 18:20, respectively. Calculation of further periodic orbits has shown that the stripes are crowded by phase-stable orbits locked in the ratio 1:1, whereas the whiskers are populated by phase-stable orbits locked in the other ratios.

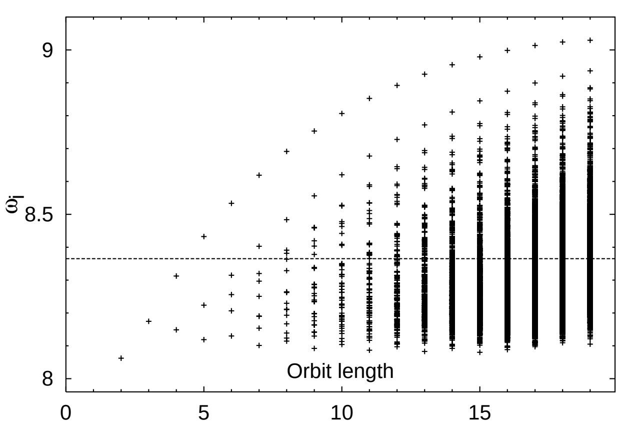

FIG. 3. Individual frequencies of unstable periodic orbits embedded into the Lorenz attractor at $r = 28$; dashed line: mean frequency of autonomous chaotic motion.

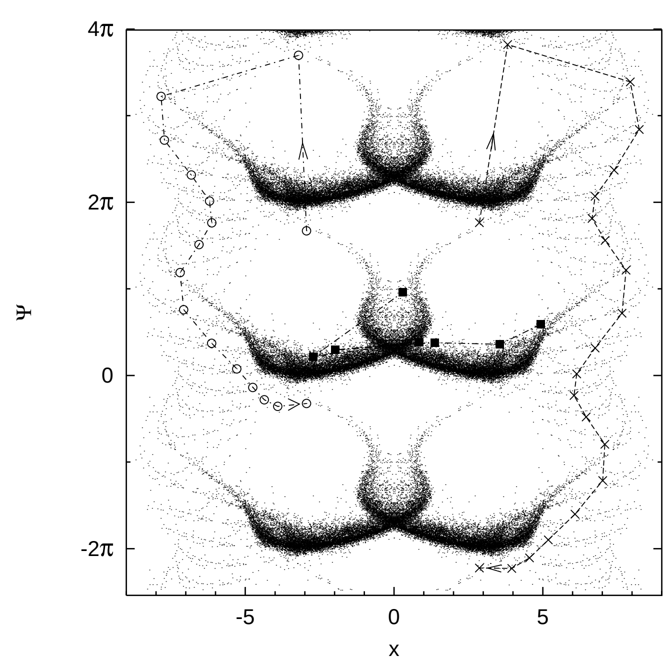

FIG. 4. Unstable periodic orbits embedded into the attractor of the forced system; $\Omega = 8.29$, $E = 6$.

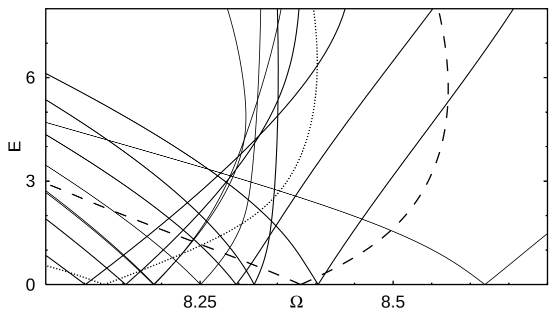

FIG. 5. Locking regions for periodic orbits; solid lines: $l = 7$ (ratio 1:1), dashed line: $l = 15$ (ratio 14:15); dotted line: $l = 20$ (ratio 18:20).

For most of the time the trajectory on the attractor wanders between unstable tori which are directly locked to the frequency of the force, but now and then it visits vicinities of tori which are locked in a ratio $(l - 1){:}l$. In such locking, the orbit of length l closes after $l - 1$ periods of the force. After the passage near such a torus the system is transported one period down along Ψ, its phase, related to the force, runs ahead by 2π, and a transition to a higher stair in the staircase of Fig. 2 occurs. To enable this, l should obey the inequality $l \geq l_0 \equiv \omega_{\max}/(\omega_{\max} - \omega_{\min})$; i.e., in our case, $l \geq 11$. (In fact, due to the broadening of Arnold tongues, l can be lower than this estimate based on the frequencies of the autonomous system.) Similarly, the phase gains of 4π proceed from rare passages near the tori locked in the frequency ratio $(l - 2){:}l$; the respective estimate $l \geq 2l_0$ yields, in our case, $l \geq 20$.

In Fig. 5 the relevant Arnold tongues for several periodic orbits are shown. In the domain of imperfect phase synchronization the short orbits (represented by nine orbits of length 7) are locked in the ratio 1:1, whereas some of the longer ones (we choose the same ones with the lengths 15 and 20 as in Fig. 4) are locked in the other ratios.

The discussed phenomenon is a "masked" form of phase synchronization: although the condition $\omega = \Omega$ is not fulfilled, dynamics remains synchronized for all the time, but synchronized to *different* frequencies. In the recent example of phase synchronization with nonconstant locking ratios in medical data [7] the switching between the ratios was apparently caused by nonstationarity of the process. Our example shows that alternation of locking ratios can happen in completely stationary deterministic systems, provided the distribution of return times is broad enough. We found the same mechanism of imperfect phase synchronization [17] by adding periodic forcing to the equations of magnetoconvection [13] and the finite-dimensional reduction of the Ginzburg-Landau equation near the boundary of the modulational instability [14].

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