Universality of the Wigner Time Delay Distribution for One-Dimensional Random Potentials

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We show that the distribution of the time delay for one-dimensional random potentials is universal in the high energy or weak disorder limit. Our analytical results are in excellent agreement with extensive numerical simulations carried out on samples whose sizes are large compared to the localization length (localized regime). The case of small samples is also discussed (ballistic regime). We provide a physical argument which explains in a quantitative way the origin of the exponential divergence of the moments. The occurrence of a log-normal tail for finite size systems is analyzed. Finally, we present exact results in the low energy limit which clearly show a departure from the universal behavior. [S0031-9007(99)09146-2]

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The problem of quantum scattering by chaotic or disordered systems is encountered in many fields ranging from atomic or molecular physics as well as in the scattering of electromagnetic microwaves. Some properties of the scattering process are well captured through the concept of time delay. This quantity, which goes back to Eisenbud and Wigner [1], is related to the time spent in the interaction region by a wave packet of energy peaked at E. It can be expressed in terms of the derivative of the S matrix with respect to the energy. In the context of chaotic scattering, the approach based on random matrix theory (RMT) provides a statistical description of the time delays. This problem was first studied by a supersymmetric approach [2] and in [3] by using a statistical analysis. This latter paper provides a derivation for the one channel case for the different universality classes. Recently it served as a starting point for [4] where the N channel distribution is shown to be given by the Laguerre ensemble of RMT. In spite of its success, such a description by RMT is not entirely satisfactory; in particular, it does not apply to strictly one-dimensional systems [5] for which strong localization effects occur. Furthermore, it does not shed much light on the physical mechanisms which are responsible for the universal distribution. In this Letter, we explore another approach by considering the scattering by a one-dimensional random potential. In this case, the existence of universal distributions was first conjectured in [6] on the basis of a comparative study of two different models. This was further supported by [7] where the random potential is still of a different kind.

The purpose of this Letter is to present a new derivation that accounts for the universality and also to provide a physical picture that explains the origin of the algebraic tail of the distribution in terms of resonances. Further details will be given elsewhere [8]. To begin, let us briefly recall the model. We consider the Schrödinger equation on the half line $x \geq 0$:

$$-\frac{d^2}{dx^2}\,\psi_k(x) + V(x)\psi_k(x) = k^2\psi_k(x)\,. \tag{1}$$

We assume that $V(x)$ has its support on the interval $[0, L]$ and impose the Dirichlet boundary condition $\psi_k(0) = 0$. Therefore, for $x \geq L$ stationary scattering states of the form

$$\psi_k(x) = \frac{1}{2}\,(e^{-\mathrm{i}k(x-L)} + e^{\mathrm{i}k(x-L)+\mathrm{i}\delta(k)}) \tag{2}$$

represent the superposition of an incoming and a reflected plane wave. Since there is only backward scattering, the reflection coefficient $e^{\mathrm{i}\delta(k)}$ is of unit modulus and the Wigner time delay takes the form $\tau(k) \stackrel{\text{def}}{=} \frac{1}{2k}\frac{d\delta(k)}{dk}$. Such a model with a random potential can be viewed as a model of a disordered sample connected to an infinite lead. Instead of using the invariant embedding method as in [9,10] or stochastic differential equations [11], our starting point is to relate the time delay to the wave function inside the sample. This may be achieved by using the identity

$$\frac{d}{dx}\left(\frac{d\psi^*}{dx}\,\frac{d\psi}{dE} - \psi^*\,\frac{d^2\psi}{dx\,dE}\right) = |\psi|^2. \tag{3}$$

By integration over $[0, L]$ one gets the so-called Smith formula [12]

$$\tau(k) = \frac{2}{k}\int_0^L dx\,|\psi_k(x)|^2 - \frac{1}{2k^2}\,\sin\delta(k)\,. \tag{4}$$

It expresses the time delay as the sum of a dwell time [13] and a term that can be neglected in the high energy limit. Inside the sample, the wave function and its derivative may be written in the form $\psi_k(x) = \mathcal{N}\,\sin\theta(x)e^{\xi(x)}$ and $\psi_k'(x) = k\mathcal{N}\,\cos\theta(x)e^{\xi(x)}$. The normalization factor $|\mathcal{N}| = e^{-\xi(L)}$ is fixed by matching the wave function at $x = L$ with the scattering states (2). We now consider the case where $V(x)$ is a random potential. In this case, the growth or decay of the envelope $e^{\xi(x)}$ of the wave function is measured by the Lyapunov exponent γ (inverse localization length $\lambda = \gamma^{-1}$). In the high energy limit, the envelope is a slow variable, while the phase $\theta(x)$ presents rapid oscillations on a scale k^{-1}. Therefore, in the high

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energy limit one can integrate over the fast variable in (4) and get

$$\tau(k) = \frac{1}{k} \int_0^L dx\, e^{2[\xi(x) - \xi(L)]}. \qquad (5)$$

This representation of the time delay holds for any realization of the disordered potential. It therefore captures all the statistical properties of $\tau(k)$ once the distribution of $\xi(x)$ is known. Denoting by x_c the correlation length of $V(x)$ and assuming that x_c and k^{-1} are the smallest length scales of the system, then it was proven in [14] that the variable $\xi(x)$ is a Brownian motion of the form $\xi(x) = \gamma x + \sqrt{\gamma}\, W(x)$, where $W(x)$ is a normalized Wiener process [$\langle W(x) \rangle = 0$, $\langle W(x) W(x') \rangle = \min(x, x')$]. Thus, the Lyapunov exponent γ controls both the drift and the fluctuations. Using the scaling properties of the Brownian motion then gives the following identity in law:

$$\tau(k) \stackrel{(\text{law})}{=} \frac{1}{k\gamma} \int_0^{\gamma L} du\, e^{-2u + 2W(u)}. \qquad (6)$$

This representation of the time delay as an exponential functional of the Brownian motion [15–17] allows one to derive a number of interesting results: (i) existence of a limit distribution (τ fixed, $L \to \infty$) with an algebraic tail [18]:

$$P(\tau) = \frac{\lambda}{2k\tau^2}\, e^{-\lambda/2k\tau}. \qquad (7)$$

(ii) Linear divergence of the first moment and exponential divergence of the higher moments [16]:

$$\langle \tau(k) \rangle = \frac{L}{k}\,, \qquad (8)$$

$$\langle \tau(k)^n \rangle = \Bigg\{ \sum_{m=2}^{n} (-1)^{n-m} C_n^m \times \frac{(m-2)!\,(2m-1)}{(n+m-1)!}\, e^{2m(m-1)L/\lambda} + \frac{(-1)^{n+1}}{n!} \left(2n \frac{L}{\lambda} + n - 1 \right) \Bigg\} \left(\frac{\lambda}{2k} \right)^n. \qquad (9)$$

(iii) Analytical expression of the probability distribution for a system of length L [see Eq. (12) of [6]]. In [6] we have shown that these results hold for two different models of random potential in the localized regime ($L \gg \lambda$).

In order to test the analytical results in the above mentioned regime it is convenient to choose a model suitable for numerical simulations. For this purpose we have considered the case where the random potential is given by a sum of delta functions of the same weight v, randomly dropped on the half line with an average density ρ (the so-called Frish and Lloyd model). [This model coincides with the Gaussian model (Halperin model) considered in [6] in the limit of a high density of impurities ($v \ll k \ll \rho$).] The equations that describe the evolution of the phase take a discrete form which can be implemented conveniently in a numerical simulation. The distribution of the time delay that we have obtained numerically in this way is in perfect agreement with Eq. (7) as soon as the high energy regime is reached. For example, we compare in Fig. 1 the analytical expression (7) with the corresponding numerical result for a regime $\rho \ll v \ll k$. The simulation was carried out for a system with 10^5 impurities of weight $v = 1$, distributed with an averaged density $\rho = 0.1$. The energy considered corresponds to $k = 10$ which is related to a localization length $\lambda = \frac{8k^2}{\rho v^2} = 8000$. The ratio $L/\lambda = 125$ is sufficiently large for the limit distribution to be reached. The numerical calculation is based on statistics of 50 000 values. It shows that the algebraic tail is well reproduced by (7) for 2000 times the typical value $\tau_{\text{typ}} = 200$. Let us stress that there is no adjustable parameter to fit the numerics. The only parameter entering in the analytical expressions is the localization length which is known for each kind of disorder.

The derivation of the statistical properties of τ given above allows one to understand the universality of the result but, on the other hand, does not shed much light on the physical mechanisms which are responsible for the occurrence of an algebraic tail. In the following, we propose a physical picture based on the existence of resonances that explains the leading exponential behavior of the moments. The starting point is to realize that the reflection of the incident wave on the random potential can in fact be viewed as a resonance tunneling process [19,20]. Indeed there exists a representation of the time delay as a superposition of resonances of energy E_α and width Γ_α in the form [21]

$$\tau(E) = 2 \sum_\alpha \frac{\Gamma_\alpha/2}{(E - E_\alpha)^2 + \Gamma_\alpha^2/4}\,. \qquad (10)$$

Obviously the dominant contribution $\tau \simeq \frac{4}{\Gamma_\alpha}$ is achieved when E is in a window of width Γ_α centered at E_α,

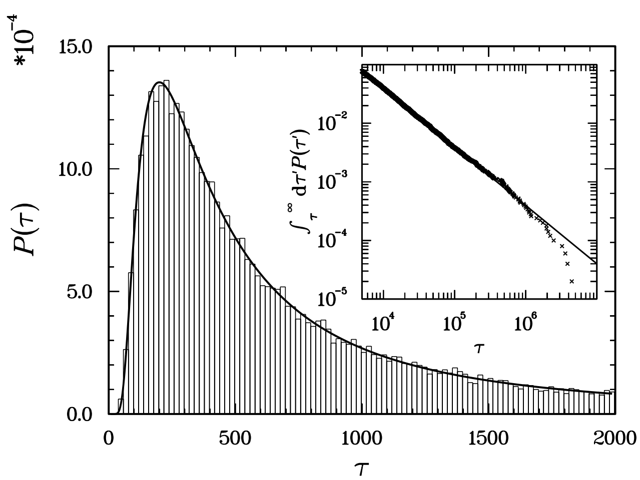

FIG. 1. Time delay distribution in the localized regime $L \gg \lambda$. Comparison between the numerical calculation and expression (7). Inset: tail of the integrated distribution, numerical and analytical.

and this will occur with probability $\frac{\Gamma}{\Delta}$, where Δ is the mean level spacing. In order to estimate the width, we may assume that a discrete level E_α localized at x_0 will be broadened by its coupling to the continuum of states through the end point $x = L$. We may therefore set $\Gamma \sim e^{-2\gamma_L(L-x_0)}$, where γ_L is the Lyapunov exponent in the finite system. Assuming that x_0 is uniformly distributed on $[0, L]$ and decorrelated from γ_L, one obtains the estimate

$$\langle \tau^n \rangle \sim \int_0^L \frac{dx_0}{L} \int d\gamma_L \, p(\gamma_L) \, \frac{\Gamma}{\Delta} \, \frac{1}{\Gamma^n} \, . \qquad (11)$$

Since $\xi(L) = L\gamma_L$ defined previously is a Gaussian process, the distribution of the finite size Lyapunov exponent γ_L is [14]

$$p(\gamma_L) = \sqrt{\frac{L}{2\pi\gamma}} \, e^{-(L/2\gamma)(\gamma_L - \gamma)^2}. \qquad (12)$$

One finally obtains

$$\langle \tau^n \rangle \sim e^{2n(n-1)L/\lambda}. \qquad (13)$$

A more refined derivation [8] allows one to recover the gross behavior of the preexponential factor. This demonstrates that this particular behavior of the moments has origin both in the exponentially small widths of the resonances and in the fluctuations of the Lyapunov exponent for the finite size sample.

The exponential divergence of the moments given by (13) resembles that of a log-normal random variable. This seems somehow paradoxical since the exact distribution $P(\tau; L)$ in the limit $L \to \infty$ (7) does not show any log-normal tail. In order to clarify this point, instead of considering as before the regime τ fixed $L \to \infty$ which leads to (7), we have studied for fixed L the tail of the distribution in the limit $\tau \to \infty$. In order to extract the asymptotic behavior, it is convenient to consider the characteristic function $\phi(p, L) = \int_0^\infty d\tau \, e^{-2kp\tau} P(\tau; L)$ given in [16]. If the conjugated variable p is chosen in a range $\gamma e^{-\gamma L} \ll p \ll \gamma e^{-\sqrt{\gamma L}}$, the characteristic function exhibits the following behavior:

$$\phi(p, L) \simeq 1 - \frac{2\sqrt{\pi} \, e^{-\gamma L/2}}{(2\gamma L)^{3/2}} \, \ln\gamma/p \left[1 + O\left(\frac{\ln\gamma/p}{\gamma L} \right) \right] \times e^{-\ln^2(\gamma/p)/8\gamma L}, \qquad (14)$$

which suggests the existence of a log-normal tail for the distribution

$$P(\tau; L) \sim \exp -\frac{1}{8\gamma L} \, \ln^2(2k\gamma\tau) \, , \qquad (15)$$

in the range $e^{\gamma L} \gg 2k\gamma\tau \gg e^{\sqrt{\gamma L}}$. Although we were not able to derive the behavior of the distribution when $e^{\gamma L} \ll 2k\gamma\tau$, the fact that the most divergent part of the moments grows like $e^{2n^2 L/\lambda}$ suggests that the distribution is still log normal.

As an aside remark, let us mention that a log-normal distribution of the time delay also occurs in the study of the random mass Dirac model at the critical point $E = 0$ [22]. There, the authors provide a representation of τ which is similar to Eq. (6), except that the drift term is absent. This problem can also be analyzed by using the approach given in [16,17].

At this stage, we have considered only a localized regime for which the size of the system is large compared to the localization length $L \gg \lambda$. Another interesting case is the ballistic one characterized by $L \ll \lambda$. In this situation, the dimensionless variable γL which arises in (6) is small compared to 1, and the argument of the exponential typically remains small compared to 1, which allows one to expand the exponential. The resulting expression for the time delay is given by a linear functional of a Gaussian quantity and has itself Gaussian fluctuations characterized by a first moment $\langle \tau \rangle = \frac{L}{k}$ and a second cumulant $\langle \tau^2 \rangle - \langle \tau \rangle^2 \simeq \frac{4\gamma}{3k^2} L^3$. We have checked numerically these results with the delta impurity model. We have considered a regime where it reproduces the high energy features of the Gaussian model: $v \ll k \ll \rho$ and $(k^2 - \rho v) \gg (\rho v^2)^{2/3}$. In this regime, one has to take into account the averaged value of the disorder ρv and replace k by $\sqrt{k^2 - \rho v}$ in all previous expressions: the localization length is thus given by $\lambda = \frac{8(k^2 - \rho v)}{\rho v^2}$ and the moments of τ now read $\langle \tau \rangle = L/\sqrt{k^2 - \rho v}$ and $\langle \tau^2 \rangle - \langle \tau \rangle^2 \simeq \frac{\rho v^2}{6(k^2 - \rho v)^2} L^3$. In Fig. 2, we compare the numerical result to the Gaussian distribution where the parameters are given by the previous expressions. The calculation is performed for a ratio $L/\lambda \simeq 1.4 \times 10^{-3}$. 10 000 values of τ were calculated.

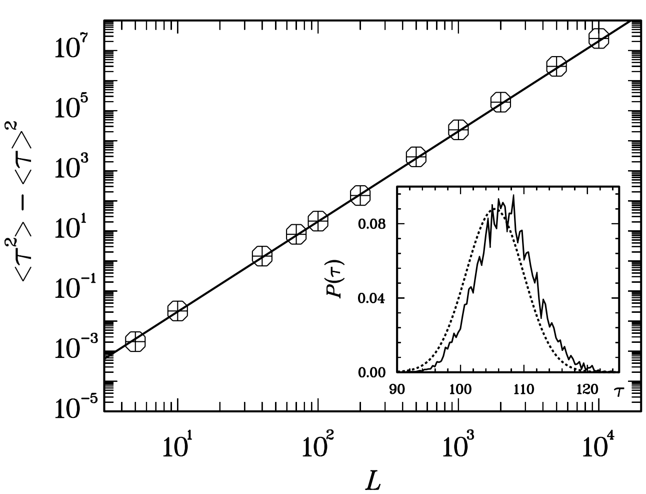

FIG. 2. Second cumulant of the time delay in the ballistic regime $L \ll \lambda$. Comparison between numerical results and the analytical result $\langle \tau^2 \rangle - \langle \tau \rangle^2 \simeq \frac{\rho v^2}{6(k^2 - \rho v)^2} L^3$. The parameters are $v = 0.001$, $\rho = 100$, and $k = 1$. Inset: time delay distribution for $L = 100$ (10^4 impurities).

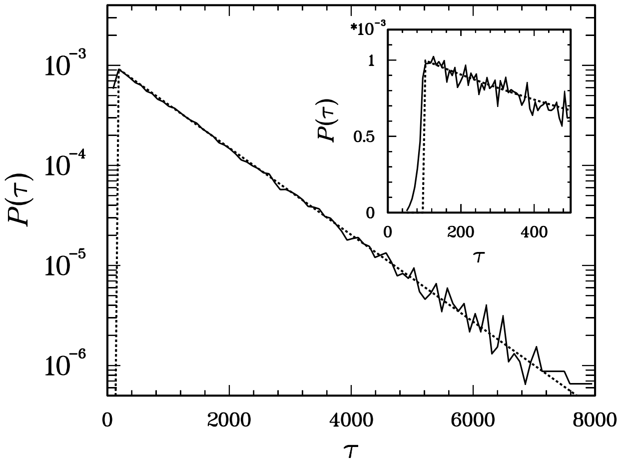

FIG. 3. Time delay distribution in the localized regime $L \gg \lambda$ at low energy.

Let us close the paper with some remarks on the low energy regime. In this case, one is more sensitive to the precise nature of the disorder, therefore one can guess that universality must break down. The distribution of the time delay will now depend on the nature of disorder. As an illustration let us consider the delta impurity model; we predict [8], in the low density regime $k \ll \rho \ll v$, an exponential tail for the distribution:

$$P(\tau) \simeq k\rho Y\left(\tau - \frac{1}{kv}\right)e^{-k\rho[\tau-(1/kv)]}, \qquad (16)$$

where $Y(x)$ denotes the Heaviside function. We have checked that this expression is in very good agreement with the numerical results. The numerical computation was performed for $v = 1$, $\rho = 0.1$, and $k = 0.01$ for 1000 impurities. The resulting distribution presented in Fig. 3 is based on 50 000 data sets. The behavior at the origin is more subtle than the one given above; nevertheless, (16) gives the correct scale on which the distribution vanishes at the origin. Another indication that universality breaks down is the fact that within this same model the low energy regime with a high density of impurities $k \ll v \ll \rho$ leads to different distributions, though still characterized by an exponential tail.

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