Kolmogorov Equation in a Fully Developed Turbulence Experiment

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The Kolmogorov equation with a forcing term is compared to experimental measurements, in low temperature helium gas, in a range of microscale Reynolds numbers R_λ between 120 and 1200. We show that the relation is accurately verified by the experiment (i.e., within $\pm 3\%$ relative error, over ranges of scales extending up to three decades). Two scales are extracted from the analysis, and revealed experimentally, one characterizing the external forcing and the other, varying as $R_\lambda^{-3/5}$, defining the position of the maximum of the function $-S_3(r)/r$, and for which a physical interpretation is offered. [S0031-9007(99)09070-5]

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The Kolmogorov equation [1] is an exact relation between the longitudinal second order and third order structure functions, $S_2(r)$ and $S_3(r)$, valid for the ideal case of homogeneous isotropic turbulence [the structure functions are defined by $S_n(r) = \langle [v(x + r) - v(x)]^n \rangle$, where v is the local velocity at x and r is a separation distance]. $S_2(r)$ is linked to kinetic energy and $S_3(r)$ is linked to energy transfers, two crucial quantities characterizing fully developed turbulence. This relation is extensively used by the experimentalists to measure, from inertial range quantities, the mean dissipation rate ϵ; no alternative method exists, in general, when the dissipative scales are unresolved, which typically happens at large Reynolds numbers. A restricted form of this equation, called the "four-fifths law," is considered one of the most important results in fully developed turbulence [2]. The Kolmogorov equation was originally derived, after von Kármán and Howarth [3], for freely decaying turbulence, and its adaptation to stationary forced turbulence, in a form suitable for a detailed comparison with experiment, was done by Novikov [4]; the corresponding equation, valid for scales well below an external forcing length L_f, reads

$$S_3 = 6\nu \frac{dS_2}{dr} - \frac{4}{5} \epsilon r \left(1 - \frac{5}{14} \frac{r^2}{L_f^2} \right), \qquad (1)$$

where r is the scale and ν is the kinematic viscosity. L_f is an external scale, characterizing the forcing. Equation (1) represents a balance between power injected at large scale, energy transfers in the inertial range, and viscous dissipation. We will call this equation a "forced Kolmogorov equation;" L_f is distinct from the integral length Λ, defined as the correlation length of the longitudinal velocity fluctuations. While the integral scale Λ is well documented, no systematic measurement of L_f has yet been reported. More generally, the extent to which the forced Kolmogorov equation describes real turbulence is poorly known. The few investigations reported thus far consider a simplified form of this equation, i.e., without the last term in the right-hand side (rhs) of Eq. (1), which, as will be shown in this Letter, is likely to be inaccurate for microscale Reynolds numbers R_λ lower than approximately 1000. For larger R_λ, the simplified form is found compatible with the experiment, but sizeable deviations, on the order of 10% to 30%, are usually observed [5–8]; the existence of such deviations raises the issue as to whether the Kolmogorov equation should be amended to apply to real systems, and whether the fundamental concepts on which it relies, i.e., isotropic homogeneous turbulence, should be reassessed. Jeopardizing these issues, the results we present in this Letter show that the forced Kolmogorov equation describes the real world to a remarkable degree of accuracy, throughout the range of scales on which it is expected to apply. The analysis will further enable us to single out two new scales for turbulence; one of them was introduced recently by Novikov [9], in a related context, but never observed.

The setup we use is the same as the one described in Refs. [10–12]. The flow is confined in a cylinder, limited axially by disks equipped with blades, rotating in opposite directions at approximately equal angular speeds. The working volume is a cylinder, 20 cm in diameter and 13.1 cm in height. The cell is enclosed in a cylindrical vessel, in thermal contact with a liquid helium bath. The vessel is filled with helium gas, held at a controlled pressure, and maintained between 4.2 and 6.5 K; the temperature is controlled with a long term stability better than 1 mK. Pressure and temperature are measured within 1% accuracy. The large scale structure of the flow is a confined circular mixing layer [10]. Local velocity measurements are performed by using "hot"-wire anemometry. The sensors are made from a 7-μm-thick carbon fiber, stretched across a rigid frame; a metallic layer covers the fiber everywhere, except on a spot at the center, 7 μm long, which defines the active length of the probes. The time responses of the probes are analyzed, in some detail, in Ref. [11]. We use here a probe located 4.7 cm from the midplane of the system and 6.5 cm from the cylinder axis; the speeds of the counter-rotating disks are finely tuned so as to maintain a local fluctuation rate close to 20%. We restrict our investigation to a range of

R_λ comprised between 120 and 1200. Here, R_λ is defined by

$$R_\lambda = \frac{u\lambda}{\nu}, \tag{2}$$

where u is the rms of the velocity fluctuations, ν is the kinematic viscosity, and $\lambda = u(15\nu/\epsilon)^{1/2}$ is the Taylor microscale (based on the measurement of ϵ discussed below).

For all data files, more than 3×10^7 data points are recorded, ensuring comfortable convergence of the second and third moments. We check, for each file, that the velocity distribution is Gaussian and we discard situations where the dissipative scale is not resolved. We also eliminate files for which the spectrum shows noise levels above 70 dB, or for which peaks (generally signaling the presence of a mechanical vibration) are visible in the inertial or dissipative ranges. This procedure leads us to reject 50% of the files. For $R_\lambda > 1200$, the noise level becomes prohibitive to ensure a reliable determination of $S_3(r)$, and for $R_\lambda > 2300$ we cease to resolve the dissipative scales.

To investigate to what extent Eq. (1) agrees with the experiment, we write it in the following form:

$$-\frac{S_3}{r} + \frac{6\nu}{r}\frac{dS_2}{dr} = \frac{4}{5}\epsilon\left(1 - \frac{5}{14}\frac{r^2}{L_f^2}\right), \tag{3}$$

following a procedure already used in Ref. [6]; we will call the expression, which forms on the left-hand side, $J(r)$. All quantities on the left-hand side can be accurately measured: The structure functions are obtained from the hot-wire time series, and viscosity is known within $\pm 2\%$ accuracy. Spatial separations r are deduced from temporal fluctuations by the use of Taylor's hypothesis (which consists of determining r by the relation $r = -Ut$, where t is the time and U is the time-averaged local velocity; it is permissible to use such a hypothesis, owing to the low level of the fluctuation rate at hand). The procedure thus consists of finding a best fit for the measured left-hand side, by using the polynomial form given by the right-hand side of Eq. (3); in this calculation, two parameters—ϵ and L_f—are free. The result is shown in Fig. 1 for $R_\lambda = 720$. One finds that the fit accurately reproduces the left-hand side of Eq. (3), within a range of scales covering two decades. The difference between the fit and experimental data is shown in the inset. The amplitudes of the deviations are below 3%, for r ranging between 8η and 900η [here, $\eta = (\nu^3/\epsilon)^{1/4}$ denotes the Kolmogorov scale]. One could say that, in this range, the forced Kolmogorov equation is verified within $\pm 3\%$ accuracy. Outside this range, discrepancies are observed: Below 8η, they are due mainly to noise which, although comfortably small for the usual measurements on turbulence, becomes here too large to be fully neglected. Above 900η, the discrepancies simply signal that the equation is no longer applicable.

Figure 2 collects results obtained for several R_λ, in Eq. (1), in the range 120–1200, i.e., one decade of variation in R_λ. We make use here of the Kolmogorov function $K(r) = -S_3/\epsilon r$ (similar to that in [6]), which represents a measure of the deviation from a constant energy transfer regime, and is thus useful for accurately analyzing the distance between the measurement and the four-fifths law limit. The points are measurements of $K(r)$ and the full lines correspond to a determination of the right-hand side of Eq. (1), using best fit values for ϵ and L_f, obtained by using the above procedure. As R_λ increases, $K(r)$ tends to form a plateau in the inertial range, as expected from the Kolmogorov theory. However, the trend, in terms of this parameter, is slow: Inspecting the set of files we have for various R_λ shows that, below $R_\lambda = 1000$, there is no clear plateau, and one may ask to what extent an inertial range can be defined below this value. Therefore, in all cases, the experiment confirms that the forced Kolmogorov relation accurately holds (the relative deviations between the theory and the experiment lying below 3%) for a set of records exploring three decades in scales, which is remarkable.

We now turn to the analysis of the dependence on R_λ of the characteristics of the curves of Fig. 2. Those curves can be characterized by several quantities. One of them, the external scale L_f, is represented at various Reynolds numbers on Fig. 3. There is substantial scatter, but no systematic evolution with R_λ is found, which indicates that L_f can be treated as a constant. We may thus consider that the effective forcing experienced by the flow is controlled by the flow geometry, which is physically acceptable. We estimate this scale as

$$L_f = 1.2 \pm 0.3 \text{ cm}. \tag{4}$$

We thus find L_f slightly smaller than the integral scale Λ (estimated to 2 cm in the present case), and 1 order

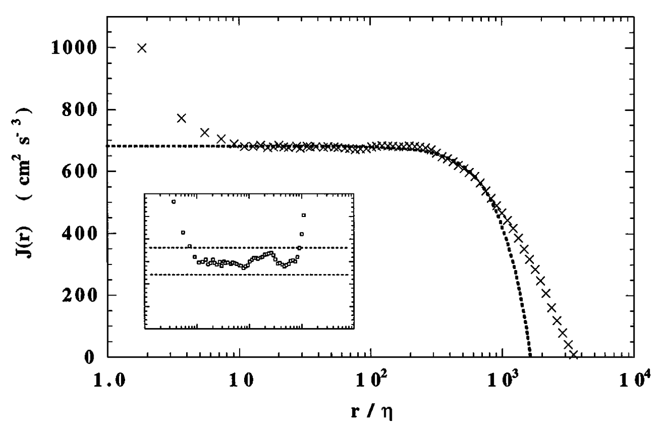

FIG. 1. ($\times$): $J(r) = -S_3/r + 6\nu S_2'/r$ compared to a best fit given by the rhs of Eq. (3), for $R_\lambda = 720$. In the inset, we show the difference between the best fit and the experiment; the full scale is $\pm 15\%$, and the dashed lines represent $\pm 3\%$. For this file, $u/U = 20.9\%$, $\eta = 11.2\ \mu$m, $L_f = 964\eta$, $l_s = 147\eta$, and $\epsilon = 855$ cm^2 s^{-3}.

of magnitude below the cell radius. The extent to which scales separate must be discussed by comparing L_f to η. The fact that L_f is 1 order of magnitude smaller than the cell size implies that, in practice, one must achieve high Reynolds numbers to obtain scale separation and therefore fill conditions in which scaling behavior can be observed. The experiment also reveals an extended gap of scales, comprised between L_f and the cell size, for which turbulent fluctuations seem to escape from a theoretical description, based on simple assumptions.

Another quantity of interest, useful for characterizing the plot of Fig. 2, is the location of the maximum of $K(r)$, which we call l_s, for reasons which will appear later. By construction, this scale is well within the inertial range. l_s is plotted against R_λ in Fig. 3; here again, there is substantial scatter, but one finds a clear power law with R_λ in the form

$$l_s = (7.1 \pm 0.6) L_f R_\lambda^{-0.57 \pm 0.04}. \qquad (5)$$

A last quantity of interest is the maximum value of $K(r)$, whose evolution with R_λ is displayed in Fig. 4. As expected, the maximum converges towards $4/5$ as R_λ increased. Nonetheless, the evolution is rather slow; the asymptotic regime is accurately reached (i.e., within 3%) only at $R_\lambda > 600$. This observation, together with the preceding remarks on the formation of a clear plateau, underlines the fact that, in experimental systems, conditions for reaching the high Reynolds number limit, for the third order structure function, are difficult to achieve. The situation seems more favorable for the even order structure functions, and for the energy spectrum, the crossover appearing at L_f being less pronounced, and hardly detectable if logarithmic scales are used.

Simple characteristics of the maximum of the Kolmogorov function $K(r)$ can be deduced from Eq. (1), by determining its approximate form, for scales well above η, in a way similar to Ref. [9]. Estimating the second order structure function S_2 by the expression

$$S_2(r) = c_0(\epsilon r)^{2/3}, \quad \text{where} \quad c_0 \simeq 2\,, \qquad (6)$$

and reinserting in Eq. (3), one gets

$$K(r) = \frac{4}{5} - 4c_0\left(\frac{r}{\eta}\right)^{-4/3} - \frac{2}{7} R_\lambda^{-3}\left(\frac{r}{\eta}\right)^2. \qquad (7)$$

Moreover, the maximum of $K(r)$ is found to be located at a scale

$$l_s \simeq L_f R_\lambda^{-3/5}, \qquad (8)$$

and the maximum of the function $K(r)$ is calculated as

$$K_{\max} = \frac{4}{5}\left[1 - \left(\frac{R_\lambda}{R_{\lambda 0}}\right)^{-6/5}\right]. \qquad (9)$$

with $R_{\lambda 0} \sim 30$. The two results are well verified in the experiment: The value $3/5$ we find is consistent with the experimental value 0.57 $\pm$0.04, and the expression for $K_{\max}$, plotted against R_λ in Fig. 4, agrees well with the experiment. We recall here that Eq. (9) applies only for moderate and large R_λ, i.e., well above 30, and should not be used for describing low Reynolds number turbulence, for which approximation (6) ceases to be valid.

It is also interesting to reveal an equivalence between l_s and another scale—which we temporarily call l_s^*—introduced by Novikov [9] quite recently, and which was proposed to represent the size of vortex strings in fully developed turbulence. The equivalence between l_s^* and l_s can be shown by reexpressing the quantity [called $\alpha(r)$] which controls, in the analysis of Ref. [9], the generation of vorticity correlations, and from which l_s^* is defined. Assuming isotropy and homogeneity, one can derive, after some manipulations, the following exact relation between $\alpha(r)$ and $S_3(r)$:

$$\alpha(r) = \frac{1}{24}\left(r\frac{d^3}{dr^3} + 8\frac{d^2}{dr^2} + \frac{8}{r}\frac{d}{dr} - \frac{8}{r^2}\right)S_3(r)\,. \qquad (10)$$

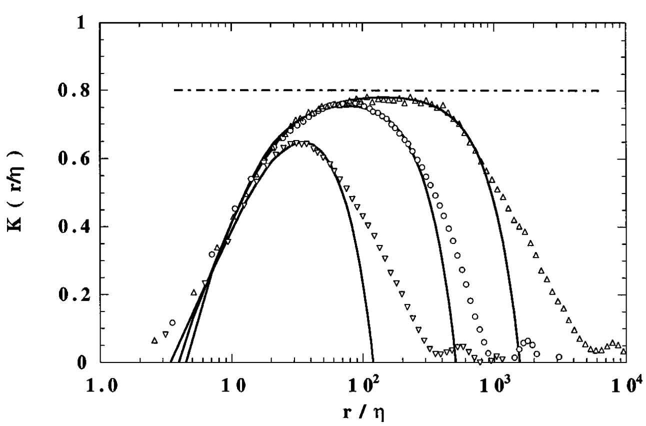

FIG. 2. The Kolmogorov function $K(r) = -S_3/\epsilon r$ versus r/η, for different Reynolds number R_λ. The values of ϵ are obtained by using best fits, as discussed in the text. (∇): $R_\lambda = 120$; ($\bigcirc$): $R_\lambda = 300$; ($\triangle$): $R_\lambda = 1170$. The solid lines show the expected curves, obtained from Eq. (1).

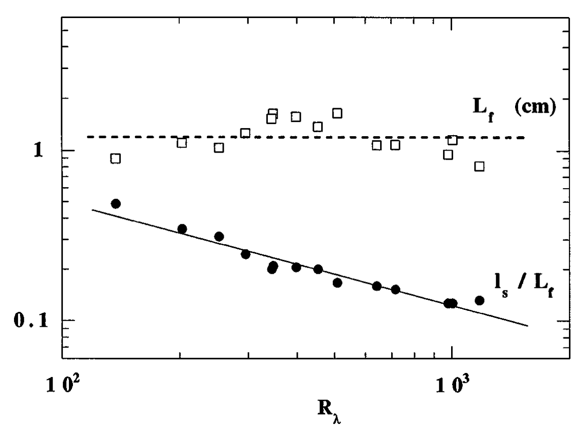

FIG. 3. Scale L_f and the ratio l_s/L_f, where l_s is defined by the extremum of $K(r)$, versus the Taylor Reynolds number R_λ. The solid line shows the best fit with an exponent -0.57 ± 0.04.

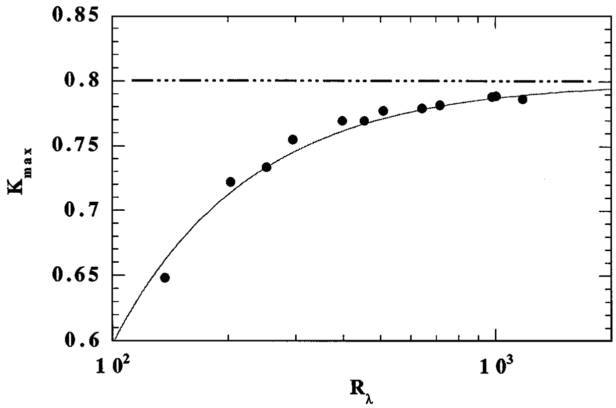

FIG. 4. Evolution of the extremum of the Kolmogorov function, $K(r) = -S_3/\epsilon r$, versus the Taylor Reynolds number R_λ. The solid line shows the best fit given by Eq. (9), with $R_{\lambda 0} \simeq 30$.

In the analysis of Ref. [9], l_s^* is defined as the zero crossing of $\alpha(r)$; now, from Eq. (10), by using the approximation leading to Eq. (7), one can show that $\alpha(r)$ crosses the zero axis at $1.4 l_s$. Therefore, the two scales are equivalent, which justifies using a single notation for both.

This remark further suggests a physical interpretation for l_s. In a previous paper [13], internally coherent clusters of worms [14,15], whose sizes are proportional to $R_\lambda^{-0.70\pm0.10}$, have been found, a scaling close to l_s. These clusters can be thought of as corresponding to the strings of Ref. [9] since they define regions where velocity gradients (assumed to represent vorticity [14]) are strongly correlated. We thus suggest here that l_s provides a scale for the worm clusters in isotropic turbulence.

In summary, we have carried out a detailed, systematic comparison between a fundamental relation of turbulence and the experiment. This study could be done by using low temperature helium, which allows one to achieve highly controlled experimental conditions while spanning an impressive range of Reynolds numbers. The analyses have been performed from single point measurements by using Taylor's hypothesis. We have shown that a forced Kolmogorov equation applies to real flows, within a range of R_λ lying between 120 and 1200, with a remarkable degree of accuracy, estimated to $\pm3\%$ in relative magnitude, over three decades of scales. The experiment suggests that, as far as the third order structure function is concerned, the formation of an inertial range is slow. However, the results we found, whether close or far from the asymptotics, can be accurately interpreted by assuming an isotropic homogeneous turbulence state, which demonstrates the relevance of this approximation, used in almost all theoretical approaches to turbulence. The scales we infer from the analysis are observed for the first time, and we suggest here that they may be useful to consider in order to characterize more completely experimental situations.

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