High Order Correlation Tensors in Turbulence

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High order correlation tensors for the spatial derivatives of the velocity field are studied with an emphasis on angular dependencies and sensitivity of these correlations to the effect of intermittency. In particular, it is found that the sum of longitudinal and transverse correlations of the deformation rates is proportional to the intermittency deviation from the classical scale similarity of velocity field. [S0031-9007(99)09207-8]

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The dynamics of turbulent motion is better understood in terms of characteristics of motion which are local in physical space and have a mechanism of amplification [1]. For three-dimensional (3D) turbulence the primary local characteristics are the vorticity field and the deformation rates [2–6]. The vorticity amplification is due to the effect of vortex stretching. For 2D turbulence the corresponding local characteristic is the vorticity gradient [6,7]. Apart from the general theory, the local characteristics of turbulence are also important in the modeling of the small-scale turbulence for the large-eddy simulations [8]. In this Letter we begin a systematic study of the high order correlation tensors for spatial derivatives of the velocity field. Special attention will be paid to the angular dependencies of such correlations and to the sensitivity to the intermittency correction.

Let us start from the correlation of the velocity increments in the inertial range (see, for example, [9]):

$$\langle u_i u_k \rangle = \langle u_r^2 \rangle \left(\frac{2+\alpha}{2}\,\delta_{ik} - \frac{\alpha}{2}\,\rho_i\rho_k \right) \tag{1}$$

$$u_i \equiv v_i' - v_i, \qquad \rho_i = r_i/r, \qquad u_r = u_i\rho_i\,, \tag{2}$$

$$\langle u_r^2 \rangle \sim r^\alpha, \qquad \alpha = \tfrac{2}{3} - \mu(\tfrac{2}{3})\,. \tag{3}$$

Here $\langle\,\rangle$ indicates statistical averaging, δ_{ij} is the unit tensor, v_i is the velocity field at point $\mathbf{x}$, prime indicates a field at point $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, and u_r is a radial (longitudinal) component of the velocity increment. The intermittency deviation from the Kolmogorov's "$\frac{2}{3}$ law" [10] is expressed in terms of the breakdown coefficients [11]. $\mu(2/3)$ has been proved to be negative [11] and is known to be small experimentally. Formula (1) is obtained for the three-dimensional locally homogeneous and isotropic turbulence with the use of the incompressibility condition:

$$\frac{\partial}{\partial r_i}\langle u_i u_k\rangle = 0\,. \tag{4}$$

From the definition (2) the two-point correlation tensors of spatial derivatives of velocity can be expressed in the form

$$\left\langle \frac{\partial^m v_i}{\partial x_{j_1}\cdots\partial x_{j_m}}\,\frac{\partial^n v_k'}{\partial x_{l_1}'\cdots\partial x_{l_n}'} \right\rangle = \frac{(-1)^{m+1}}{2}\,\frac{\partial^{m+n}}{\partial r_{j_1}\cdots\partial r_{l_n}} \times \langle u_i u_k\rangle\,. \tag{5}$$

For the corresponding spectral tensor we get the simple formula

$$(-1)^m (i)^{m+n}\,\frac{E(p)}{4\pi p^2}\,p_{j_1}\cdots p_{l_n}(\delta_{ik} - p_i p_k p^{-2})\,, \tag{6}$$

where $E(p)$ is the energy spectrum and in the inertial range $E(p) \sim p^{-1-\alpha}$. The spatial structure of the tensor (5) is more complex and physically revealing, as we will see below.

In this Letter we consider the fourth order tensor with $m = n = 1$. Substitution of (1) and (3) into (5), after some algebra, gives

$$\begin{aligned}\left\langle \frac{\partial v_i}{\partial x_j}\,\frac{\partial v_k'}{\partial x_l'} \right\rangle = q(r)[&(2+\alpha)\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} - (4-\alpha^2)\delta_{ik}\rho_j\rho_l \\ &+ (2-\alpha)(\delta_{ij}\rho_k\rho_l + \delta_{il}\rho_j\rho_k + \delta_{jk}\rho_i\rho_l + \delta_{jl}\rho_i\rho_k + \delta_{kl}\rho_i\rho_j) \\ &- (4-\alpha)(2-\alpha)\rho_i\rho_j\rho_k\rho_l]\,,\end{aligned} \tag{7}$$

where

$$q(r) = \frac{\alpha\langle u_r^2\rangle}{4r^2}\,. \tag{8}$$

Using definitions of vorticity $\omega_i = \varepsilon_{ijk}\partial v_k/\partial x_j$ (ε_{ijk} is the unit antisymmetric tensor) and deformation rates $D_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, from (7) we get

$$\langle \omega_i D_{kl}' \rangle = -\frac{(3+\alpha)(2-\alpha)}{2}\,q(r) \times (\varepsilon_{ijk}\rho_j\rho_l + \varepsilon_{ijl}\rho_j\rho_k)\,. \tag{9}$$

This tensor is determined by only one scalar and its structure is universal. The α dependence of the scalar is

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such that the intermittency correction is not significant. A more general correlation tensor

$$\left\langle \omega_i \frac{\partial v_k'}{\partial x_l'} \right\rangle = -(3+\alpha)q(r)[\varepsilon_{ikl} + (2-\alpha)\varepsilon_{ijk}\rho_j\rho_l] \quad (10)$$

has two scalars and its structure depends on α, but again the scalars are not α sensitive. A similar situation is for the vorticity correlation tensor, as follows from (7):

$$\langle \omega_i \omega_k' \rangle = (3+\alpha)q(r)[\alpha\delta_{ik} + (2-\alpha)\rho_i\rho_k]. \quad (11)$$

Let us consider correlation for one vorticity component ($i = k = 1$):

$$\langle \omega_1 \omega_1' \rangle = (3+\alpha)q(r)[\alpha + (2-\alpha)\cos^2\phi], \quad \rho_1 = \cos\phi, \quad (12)$$

where ϕ is the angle between direction of vorticity and $\mathbf{r}$. We see that this correlation is always positive (in the inertial range) and the longitudinal correlation ($\phi = 0$) is $2/\alpha \approx 3$ times bigger than the transversal ($\phi = \pi/2$). Thus, even on the level of second order vorticity correlation, we see a tendency for formation of coherent vortex filaments in 3D turbulent flows. We note that analysis of the third order vortex correlations [5] leads to the "vortex strings" scale $l_s \sim L \times \mathrm{Re}^{-3/10}$ inside the inertial range (L is the external scale; Re is the Reynolds number). This scaling has now an experimental support [12,13].

Now we turn to the deformation rates. Symmetrization of (7) gives

$$\begin{aligned}\langle D_{ij}D_{kl}' \rangle = \frac{q(r)}{4}[&2(1+\alpha)(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) - 4\delta_{ij}\delta_{kl} \\ &+ (2-\alpha)(1-\alpha)(\delta_{ik}\rho_j\rho_l + \delta_{jk}\rho_i\rho_l + \delta_{il}\rho_j\rho_k + \delta_{jl}\rho_i\rho_k) \\ &+ 4(2-\alpha)(\delta_{ij}\rho_k\rho_l + \delta_{kl}\rho_i\rho_j) - 4(4-\alpha)(2-\alpha)\rho_i\rho_j\rho_k\rho_l]. \end{aligned} \quad (13)$$

Consider correlation for one component of deformation $D_{11} = \partial v_1/\partial x_1$. Formula (13) [or (7)] gives

$$D(r,\alpha,\phi) \equiv \langle D_{11}D_{11}' \rangle = q(r)f(\alpha,\phi), \quad (14)$$

$$f(\alpha,\phi) = \alpha + (3-\alpha)(2-\alpha)\cos^2\phi - (4-\alpha)(2-\alpha)\cos^4\phi, \quad (15)$$

where, as before, $\rho_1 = \cos\phi$, $q(r)$ depends on α implicitly [see (3), (8)]. We see that correlation (14) changes sign: $f(\alpha,0) = -2(1-\alpha)$, $f(\alpha,\pi/2) = \alpha$. It has maximum $f_m(\alpha) = \alpha + (2-\alpha)(3-\alpha)^2/4(4-\alpha)$ with $\cos^2\phi_m = (3-\alpha)/2(4-\alpha)$, $\phi_m \approx 54°$. The most remarkable property of this correlation is the following relation:

$$D(r,\alpha,0) + D(r,\alpha,\pi/2) = -3\mu(\tfrac{2}{3})q(r). \quad (16)$$

It means that the sum of longitudinal and transverse correlations is proportional to the intermittency correction. Thus, we found a correlation characteristic of velocity gradients which is very sensitive to the effect of intermittency.

We note that all presented results are exact consequences of incompressibility, power dependence (3), and local isotropy in the inertial range of the well developed 3D turbulence. The only mathematical procedure we used in this Letter is spatial differentiation of formula (1). Content underlaying this simple procedure ("differentiate and reign") is a transition to local characteristics. We can recommend this procedure for other areas of physics, where local characteristics can be identified (a technique for such identification with use of conditional averaging is presented in Ref. [6]).

We hope that these results will stimulate another direction in analytical, experimental, and numerical study of turbulence. Particularly, formula (16) presents a natural way to measure the intermittency correction, which is the subject of controversy in the literature.

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