Harmonic Measure Exponents for Two-Dimensional Percolation

Bertrand Duplantier
Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette Cedex, France
and Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France
(Received 23 November 1998)

The harmonic measure (or diffusion field) near a critical percolation cluster in two dimensions (2D) is considered. Its moments, summed over the accessible external hull, exhibit a multifractal (MF) spectrum, which I calculate exactly. The generalized dimensions $D(n)$ as well as the MF function $f(\alpha)$ are derived from generalized conformal invariance, and are shown to be identical to those of the harmonic measure on 2D random or self-avoiding walks. An application to the impedance of a rough percolative electrode is given. The numerical checks are excellent. [S0031-9007(99)09126-7]

PACS numbers: 05.45.Df, 05.40.Jc, 41.20.Cv

Percolation theory, whose tenuous fractal structures, called incipient clusters, present fascinating properties, has served as an archetypal model for critical phenomena [1]. The subject has recently enjoyed renewed interest: the scaling (continuum) limit has fundamental properties, e.g., conformal invariance, which present a mathematical challenge [2–4]. Almost uncharted territory in exact fractal studies is the *harmonic measure*, i.e., the diffusion or electrostatic field near an equipotential fractal boundary, whose self-similarity is reflected in a *multifractal* (MF) behavior of the harmonic measure [5].

MF exponents for the harmonic measure of fractals are especially important in two contexts: diffusion-limited aggregation (DLA) and the double layer impedance at a surface. In DLA, the harmonic measure actually determines the growth process and its scaling properties are intimately related to those of the cluster itself [6]. The double layer impedance at a rough surface between a good conductor and an ionic medium presents an anomalous frequency dependence, which has been observed by electrochemists for decades. It was recently proposed that this is at heart a multifractal phenomenon, directly linked with the harmonic measure of the rough electrode [7]. In both of the preceding contexts, percolation clusters have been studied numerically as generic models.

In this Letter, I consider incipient percolation clusters in two dimensions (2D), and determine analytically the exact multifractal exponents of their harmonic measure. I use recent advances in conformal invariance (linked to quantum gravity), which allow for the mathematical description of random walks interacting with other random fractal structures, such as random walks [8,9], and self-avoiding walks [10]. A difficulty here is the presence of a subtle fjord structure in the percolation cluster hull, which has only recently been understood [11]. Excellent agreement with decade-old numerical data is obtained, thereby confirming the relevance of conformal invariance to multifractality; the exact prediction for the anomalous exponent of a percolative electrode also corroborates its multifractal nature.

Consider a single, isolated, two-dimensional very large incipient cluster C, at the percolation threshold p_c. Define $H(w)$ as the probability that a random walker (RW) launched from infinity *first* hits the outer (accessible) percolation hull $\mathcal{H}(C)$ at point $w \in \mathcal{H}(C)$. We are especially interested in the moments of H, averaged over all realizations of RW's and C

$$\mathcal{Z}_n = \left\langle \sum_{w \in \mathcal{H}} H^n(w) \right\rangle, \tag{1}$$

where n can be, *a priori*, a real number. For very large clusters C and hulls $\mathcal{H}(C)$ of average size R, one expects these moments to scale as

$$\mathcal{Z}_n \approx (a/R)^{\tau(n)}, \tag{2}$$

where a is a microscopic cutoff, and where the multifractal scaling exponents $\tau(n)$ encode generalized dimensions $D(n)$, $\tau(n) = (n-1)D(n)$, which vary in a nonlinear way with n [12–15]. Several *a priori* results are known. $D(0)$ is the Hausdorff dimension of the support of the measure. By construction, H is a normalized probability measure, so that $\tau(1) = 0$. Makarov's theorem [16], here applied to the Hölder regular curve describing the hull [17], gives the *nontrivial* information dimension $\tau'(1) = D(1) = 1$. The multifractal formalism [12–15] further involves characterizing subsets $\mathcal{H}_\alpha$ of sites of the hull $\mathcal{H}$ by a Hölder exponent α, such that their local H measure in a ball of radius a scales as $H(w \in \mathcal{H}_\alpha) \approx (a/R)^\alpha$. The "fractal dimension" $f(\alpha)$ of the set $\mathcal{H}_\alpha$ is given by the symmetric Legendre transform of $\tau(n)$,

$$\alpha = \frac{d\tau}{dn}(n), \qquad \tau(n) + f(\alpha) = \alpha n,$$
$$n = \frac{df}{d\alpha}(\alpha). \tag{3}$$

Because of the ensemble average (1), values of $f(\alpha)$ can become negative for some domains of α [18].

This Letter is organized as follows: I first present in detail the findings and their potential physical significance and applications, before proceeding with the more abstract mathematical derivation.

 0031-9007/99/82(20)/3940(4)$15.00

My results for the generalized harmonic dimensions for percolation are

$$D(n) = \frac{1}{2} + \frac{5}{\sqrt{24n+1}+5}, \qquad n \in [-\tfrac{1}{24}, +\infty), \tag{4}$$

valid for all values of moment order $n, n \geq -\frac{1}{24}$. The Legendre transform (3) of $\tau(n) = (n-1)D(n)$ reads

$$\alpha = \frac{d\tau}{dn}(n) = \frac{1}{2} + \frac{5}{2}\frac{1}{\sqrt{24n+1}}, \tag{5}$$

and

$$f(\alpha) = \frac{25}{48}\left(3 - \frac{1}{2\alpha - 1}\right) - \frac{\alpha}{24}, \qquad \alpha \in (\tfrac{1}{2}, +\infty). \tag{6}$$

Figure 1 shows the exact curve $D(n)$ (4) together with the numerical results for $n \in \{2, \ldots, 9\}$ by Meakin *et al.* [19], showing fairly good agreement.

The first striking observation is that the dimension of the support of the measure $D(0) \neq D_H$, where $D_H = \frac{7}{4}$ is the Hausdorff dimension of the standard hull, i.e., the outer boundary of critical percolating clusters [20]. In fact, the value $D_{EP} \equiv D(0) = \frac{4}{3}$ corresponds to the *accessible external perimeter* [21], the other hull sites being located in deep fjords, which are not probed by the harmonic measure. This structure and the exact value of D_{EP} are elucidated in terms of path-crossing statistics in Aizenman *et al.* [11]. In the *scaling continuous* regime, the fjords *do* close, yielding a *smoother* (self-avoiding) accessible perimeter of dimension $\frac{4}{3}$. This agrees with the original instability phenomenon observed numerically on a lattice [21].

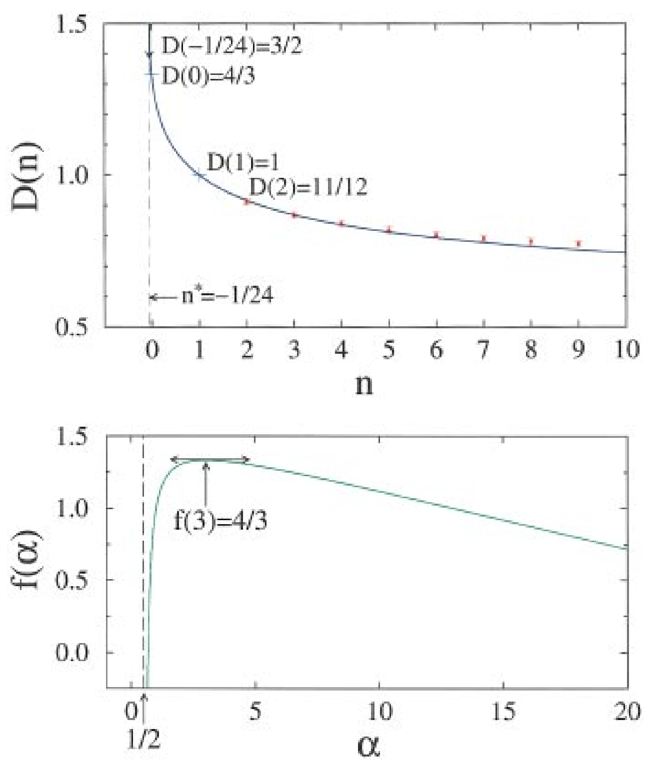

FIG. 1(color). Universal harmonic multifractal dimensions $D(n)$, and spectrum $f(\alpha)$ of a 2D incipient percolation cluster, compared to numerical results by Meakin *et al.* [19] (in red).

An even more striking fact is the complete identity of Eqs. (4)–(6) to the corresponding results *both* for random walks and for self-avoiding walks (SAW's) [10]. In particular, $D(0) = \frac{4}{3}$ is the Hausdorff dimension of a SAW, common to the *external frontier* of a percolation hull and of a Brownian motion [8,9]. Seen from outside, these three fractal curves are not distinguished by the harmonic measure. As we shall see, this fact is linked to the presence of a universal underlying conformal field theory with a vanishing central charge $c = 0$. In other respects, a 2D polymer at the Θ *point* is known to obey exactly the statistics of a percolation hull [22], and the MF results (4)–(6) therefore apply *also* to that case.

The minimal value of α $(= \frac{1}{2})$ is due to the strongest singularity of the harmonic measure, i.e., near a needle. The linear asymptote of the $f(\alpha)$ curve for $\alpha \to +\infty$, $f(\alpha) \sim -\frac{\alpha}{24}$, corresponds to the lowest part $n \to n^* = -\frac{1}{24}$ of the spectrum of dimensions. Its linear shape is quite reminiscent of the case of a 2D DLA cluster [23]. Define $\mathcal{N}(H)$ as the number of sites having a probability H to be hit. Using the MF formalism to change from variable H to α (at fixed value of a/R) shows that $\mathcal{N}(H)$ obeys, for $H \to 0$, a power law behavior with an exponent $\tau^* = 1 + \lim_{\alpha \to +\infty} \frac{1}{\alpha} f(\alpha) = 1 + n^*$. Thus we predict

$$\mathcal{N}(H)|_{H \to 0} \approx H^{-\tau^*}, \qquad \tau^* = \frac{23}{24}. \tag{7}$$

This $\tau^* = 0.958\,33\ldots$ compares very well with the numerical result $\tau^* = 0.951 \pm 0.030$, obtained for $10^{-5} \leq H \leq 10^{-4}$ [19].

Let us consider for a moment the different, but related, problem of the *double layer impedance* of a *rough* electrode. In some range of frequencies ω, the impedance contains an anomalous "constant phase angle" (CPA) term $(i\omega)^{-\beta}$, where $\beta < 1$. From a natural RW representation of the impedance, a scaling law was recently proposed: $\beta = \frac{D(2)}{D(0)}$ (here in 2D), where $D(2)$ and $D(0)$ are the multifractal dimensions of the H measure on the rough electrode [7]. In the case of a 2D porous percolative electrode, our results (4) give $D(2) \equiv \frac{11}{12}$, $D(0) = \frac{4}{3}$, whence $\beta = \frac{11}{16} = 0.6875$. This compares very well with a numerical RW algorithm result [24], which yields an effective CPA exponent $\beta \simeq 0.69$, nicely vindicating the multifractal description [7].

Let me now give the main lines of the derivation of exponents $D(n)$ by generalized *conformal invariance.* We focus on site percolation on the 2D triangular lattice; by universality the results are expected to apply to other 2D (e.g., bond) percolation models. The boundary lines of the percolation clusters, i.e., of connected sets of occupied hexagons, form self-avoiding lines on the dual hexagonal lattice.

By the very definition of the H measure, n independent RW's diffusing away from the hull give a geometric representation of the nth moment H^n, for *n integer*. The values so derived for $n \in \mathbb{N}$ will be enough, by convexity arguments, to obtain the analytic continuation for arbitrary n's. Figure 2 depicts n independent random walks, in a bunch, *first* hitting the external hull of a percolation cluster at a site $w = (\bullet)$.

The bunch of independent RW's avoids the occupied cluster, and defines its own envelope as a set of two *boundary* lines separating it from the occupied part of the lattice. The site $(\bullet)$, to belong to the *accessible* hull, thus remains, in the *continuous scaling limit*, the source of at least *three nonintersecting crossing paths*, noted S_3, reaching to a (large) distance R [11]. These (self-avoiding) paths are *monochromatic*: one path runs only through occupied (light blue) sites; the other two, *dual* lines, run through empty (white) sites, in between the accessible cluster and RW's frontiers (Fig. 2). The definition of the *standard* hull requires only the origination, in the scaling limit, of a "bichromatic" pair of lines S_2. Points lacking the second dual line are not accessible to

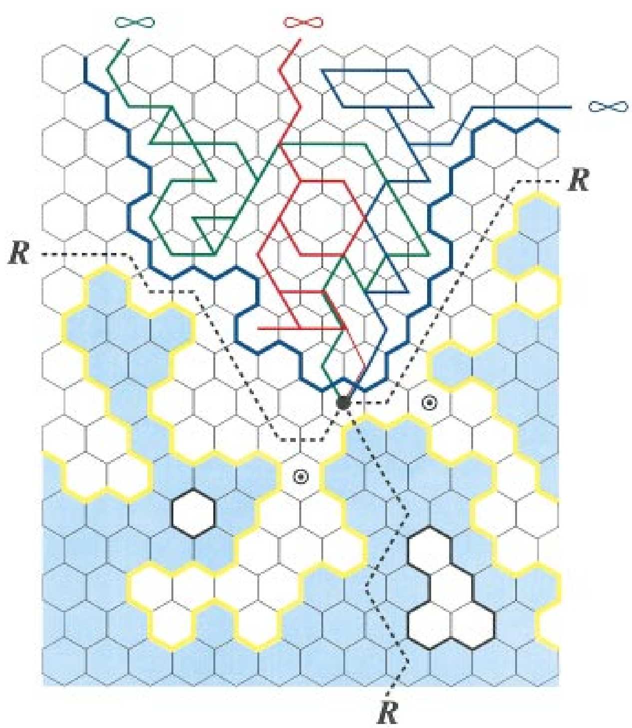

FIG. 2(color). An "active" site $(\bullet)$ on the accessible external perimeter for site percolation on the triangular lattice. It is defined by the existence, in the *scaling limit*, of $\ell = 3$ nonintersecting crossing paths S_3 (dotted lines), one on the incipient (light blue) cluster, the other two on the dual empty (white) sites. The points $\odot$ are entrances of fjords, which close in the scaling limit and will not support the harmonic measure. Point $(\bullet)$ is first hit by three independent RW's (red, green, blue), contributing to $H^3(\bullet)$. The hull of the incipient cluster (golden line) avoids the outer frontier of the RW's (thick blue line). A Riemann map of the latter onto the real line $\mathbb{R}$ reveals the presence of an underlying $\ell = 3$ path-crossing *boundary* operator, i.e, a two-cluster boundary operator, with dimension in the half-plane $\tilde{x}_{\ell=3} = \tilde{x}^C_{k=2} = 2$. Both accessible hull and Brownian paths have a frontier dimension $\frac{4}{3}$.

RW's after the scaling limit is taken, because their (white) exit path becomes a strait pinched by other parts of the (light blue) occupied cluster.

Let us introduce the notation $A \wedge B$ for two sets, A and B, of random paths, conditioned to be *mutually avoiding*, and $A \vee B$ for two *independent*, thus possibly intersecting, sets [10]. Now consider n independent RW's, or Brownian paths $\mathcal{B}$ in the scaling limit, in a bunch noted $(\vee\mathcal{B})^n$, *avoiding* a set $S_\ell \equiv (\wedge\mathcal{P})^\ell$ of ℓ *nonintersecting* and self-avoiding crossing paths in the percolation system. They originate from the same hull site, and each passes only through occupied sites, or only through empty (*dual*) ones [11]. The probability that the Brownian and percolation paths altogether traverse the annulus $\mathcal{D}(a,R)$ from the inner boundary circle of radius a to the outer one at distance R, i.e., are in a "star" configuration $S_\ell \wedge (\vee\mathcal{B})^n$ (Fig. 2), is expected to scale for $R/a \to \infty$ as

$$\mathcal{P}_R(S_\ell \wedge n) \approx (a/R)^{x(S_\ell \wedge n)}, \tag{8}$$

where we used $S_\ell \wedge n \equiv S_\ell \wedge (\vee\mathcal{B})^n$ as a shorthand notation, and where $x(S_\ell \wedge n)$ is a new critical exponent depending on ℓ and n. It is convenient to introduce similar *surface* probabilities $\tilde{\mathcal{P}}_R(S_\ell \wedge n) \approx (a/R)^{\tilde{x}(S_\ell \wedge n)}$ for the same star configuration of paths, now crossing through the half-annulus $\tilde{\mathcal{D}}(a,R)$ in the *half-plane*.

When $n \to 0, \mathcal{P}_R(S_\ell)$ $(\tilde{\mathcal{P}}_R(S_\ell))$ is the probability of having ℓ simultaneous nonintersecting path crossings of the annulus in the plane (half-plane), with associated exponents $x_\ell \equiv x(S_\ell \wedge 0)$ and $\tilde{x}_\ell \equiv \tilde{x}(S_\ell \wedge 0)$ [11]. In terms of probability (8), the harmonic measure moments (1) and (2) simply scale as $\mathcal{Z}_n \approx R^2 \mathcal{P}_R(S_{\ell=3} \wedge n)$ [18], which leads to

$$\tau(n) = x(S_3 \wedge n) - 2\,. \tag{9}$$

Using the fundamental mapping of the conformal field theory (CFT) in the *plane* $\mathbb{R}^2$, describing a critical statistical geometrical system, to the CFT on a fluctuating abstract random Riemann surface, i.e., in the presence of *quantum gravity* [25], I have recently shown that there exist two universal functions, U and V, depending only on the central charge c of the CFT, which suffice to generate all geometrical exponents involving *mutual avoidance* of random *star-shaped* sets of paths of the critical system [10]. For $c = 0$, which corresponds to RW's, SAW's, and *percolation*, these universal functions are

$$U(x) = \tfrac{1}{3}\,x(1 + 2x), \qquad V(x) = \tfrac{1}{24}\,(4x^2 - 1)\,, \tag{10}$$

with $V(x) \equiv U(\frac{1}{2}(x - \frac{1}{2}))$. Consider now two arbitrary random sets A and B, involving each a collection of paths in a star configuration, with proper scaling crossing exponents $x(A), x(B)$, or, in the half-plane, crossing exponents $\tilde{x}(A), \tilde{x}(B)$. If one fuses the star centers and requires A and B to stay mutually avoiding, then the new crossing exponents, $x(A \wedge B)$ and $\tilde{x}(A \wedge B)$, obey the *star algebra* [8,10]

$$\begin{aligned} x(A \wedge B) &= 2V[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))],\\ \tilde{x}(A \wedge B) &= U[U^{-1}(\tilde{x}(A)) + U^{-1}(\tilde{x}(B))], \end{aligned} \tag{11}$$

where $U^{-1}(x)$ is the inverse function of U

$$U^{-1}(x) = \frac{1}{4}(\sqrt{24x + 1} - 1). \tag{12}$$

If, on the contrary, A and B are *independent* and can overlap, then by trivial factorization of probabilities, $x(A \vee B) = x(A) + x(B)$, and $\tilde{x}(A \vee B) = \tilde{x}(A) + \tilde{x}(B)$ [10]. The rules (11), which mix bulk and boundary exponents, can be understood as simple factorization properties on a random Riemann surface, i.e., in quantum gravity [8,10], or as recurrence relations in $\mathbb{R}^2$ between conformal Riemann maps of the successive mutually avoiding paths onto the line $\mathbb{R}$ [9]. On a random surface, $U^{-1}(\tilde{x})$ is the boundary dimension corresponding to the value $\tilde{x}$ in $\mathbb{R} \times \mathbb{R}^+$, and the sum of U^{-1} functions in Eq. (11) represents linearly the juxtaposition $A \wedge B$ of two sets of random paths near their random frontier, i.e., the product of two "boundary operators" on the random surface. The latter sum is mapped by the functions U and V into the scaling dimensions in $\mathbb{R}^2$ [10]. The structure thus unveiled is so stringent that it immediately gives both the percolation crossing exponents x_ℓ and $\tilde{x}_\ell$ [11], and our harmonic measure exponents $x(S_\ell \wedge n)$ (8). First, for a set $S_\ell = (\wedge \mathcal{P})^\ell$ of ℓ crossing paths, we have from the recurrent use of (11)

$$x_\ell = 2V[\ell U^{-1}(\tilde{x}_1)], \qquad \tilde{x}_\ell = U[\ell U^{-1}(\tilde{x}_1)]. \tag{13}$$

For percolation, two values of half-plane crossing exponents $\tilde{x}_\ell$ are known by *elementary* means: $\tilde{x}_2 = 1, \tilde{x}_3 = 2$ [3,11]. From (13) we thus find $U^{-1}(\tilde{x}_1) = \frac{1}{2}U^{-1}(\tilde{x}_2) = \frac{1}{3}U^{-1}(\tilde{x}_3) = \frac{1}{2}$ (thus $\tilde{x}_1 = \frac{1}{3}$ [26]), which in turn gives

$$x_\ell = 2V(\tfrac{1}{2}\ell) = \frac{1}{12}(\ell^2 - 1),$$

$$\tilde{x}_\ell = U(\tfrac{1}{2}\ell) = \frac{\ell}{6}(\ell + 1).$$

We thus recover the identity [11] $x_\ell = x_{L=\ell}^{\mathcal{O}(N=1)}, \tilde{x}_\ell = \tilde{x}_{L=\ell+1}^{\mathcal{O}(N=1)}$ with the L-line exponents of the associated $\mathcal{O}(N=1)$ loop model, in the "low-temperature phase." For L *even*, these exponents also govern the existence of $k = \frac{1}{2}L$ *spanning* clusters, with the identity $x_k^C = x_{\ell=2k} = \frac{1}{12}(4k^2 - 1)$ in the bulk, and $\tilde{x}_k^C = \tilde{x}_{\ell=2k-1} = \frac{1}{3}k(2k-1)$ in the half-plane [20,27]. The nonintersection exponents of k' random walks are also given by x_ℓ, $\tilde{x}_\ell$ for $\ell = 2k'$ [8], so we obtain a *complete* equivalence between a Brownian path and *two* percolating crossing paths, in both the plane and half-plane.

Finally, for the harmonic exponents in (8), we fuse the two objects S_ℓ and $(\vee \mathcal{B})^n$ into a new star $S_\ell \wedge n$ (see Fig. 2), and use (11). We just have seen that the boundary ℓ-crossing exponent of S_ℓ, $\tilde{x}_\ell$, obeys $U^{-1}(\tilde{x}_\ell) = \frac{1}{2}\ell$. The bunch of n independent Brownian paths have their own half-plane crossing exponent $\tilde{x}((\vee \mathcal{B})^n) = n\tilde{x}(\mathcal{B}) = n$, since the boundary dimension of a single Brownian path is trivially $\tilde{x}(\mathcal{B}) = 1$ [8]. Thus we obtain

$$x(S_\ell \wedge n) = 2V(\tfrac{1}{2}\ell + U^{-1}(n)). \tag{14}$$

Specifying to the case $\ell = 3$ finally gives from (10) and (12)

$$x(S_3 \wedge n) = 2 + \tfrac{1}{2}(n-1) + \tfrac{5}{24}(\sqrt{24n + 1} - 5),$$

from which $\tau(n)$, Eq. (9), and $D(n)$, Eq. (4), follow, **QED**.

I thank M. Aizenman, D. Kosower, and T. C. Halsey for fruitful discussions.

[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992).

[2] R. Langlands, P. Pouliot, and Y. Saint-Aubin, Bull. Am. Math. Soc. **30**, 1 (1994); J. L. Cardy, J. Phys. A **25**, L201 (1992).

[3] M. Aizenman, in *Mathematics of Multiscale Materials*, the IMA Volumes in Mathematics, edited by K. M. Golden *et al.* (Springer-Verlag, Berlin, 1998), Vol. 99.

[4] I. Benjamini and O. Schramm, Commun. Math. Phys. **197**, 75 (1998).

[5] B. B. Mandelbrot and C. J. G. Evertsz, Nature (London) **348**, 143 (1990).

[6] T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[7] T. C. Halsey and M. Leibig, Ann. Phys. (N.Y.) **219**, 109 (1992).

[8] B. Duplantier, Phys. Rev. Lett. **81**, 5489 (1998).

[9] G. F. Lawler and W. Werner (to be published).

[10] B. Duplantier, Phys. Rev. Lett. **82**, 880 (1999).

[11] M. Aizenman, B. Duplantier, and A. Aharony (to be published).

[12] B. B. Mandelbrot, J. Fluid. Mech. **62**, 331 (1974).

[13] H. G. E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 835 (1983).

[14] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, Proceedings of the International School of Physics "Enrico Fermi," Course LXXXVIII, edited by M. Ghil (North-Holland, New York, 1985), p. 84.

[15] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[16] N. G. Makarov, Proc. London Math. Soc. **51**, 369 (1985).

[17] M. Aizenman and A. Burchard, Duke Math. J. (to be published).

[18] M. E. Cates and T. A. Witten, Phys. Rev. A **35**, 1809 (1987).

[19] P. Meakin *et al.*, Phys. Rev. A **34**, 3325 (1986); see also P. Meakin, *ibid.* **33**, 1365 (1986); in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic, London, 1988), Vol. 12.

[20] H. Saleur and B. Duplantier, Phys. Rev. Lett. **58**, 2325 (1987).

[21] T. Grossman and A. Aharony, J. Phys. A **20**, L1193 (1987).

[22] B. Duplantier and H. Saleur, Phys. Rev. Lett. **59**, 539 (1987).

[23] R. C. Ball and R. Blumenfeld, Phys. Rev. A **44**, R828 (1991).

[24] P. Meakin and B. Sapoval, Phys. Rev. A **46**, 1022 (1992).

[25] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[26] J. L. Cardy, Nucl. Phys. **B240**, 514 (1984).

[27] B. Duplantier, Phys. Rep. **184**, 229 (1989).