# Algebraic Approach in Unifying Quantum Integrable Models

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A novel algebra underlying integrable systems is shown to generate and unify a large class of quantum integrable models with given $R$ matrix, through reductions of an ancestor Lax operator and its different realizations. Along with known discrete and field models a new class of inhomogeneous and impurity models is obtained. [S0031-9007(99)09121-8]

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The self-dual Yang-Mills equation with possible reductions has given a vivid unifying picture in classical integrable systems in $1 + 1$ and $0 + 1$ dimensions [1]. However, in the quantum case not much has been achieved in this direction, and there exists a genuine need for discovering some scheme which would generate models of quantum integrable systems (QIS) [2] along with their Lax operators and $R$ matrix and thus unifying them.

The significance of algebraic structures in describing physical consequences is well recognized. Like Lie algebras, their quantum deformations [3] are also found to be of immense importance in physical models [4–6]. In fact, the idea of quantum Lie algebra which attracted enormous interest in recent years [7–9], has stemmed from the QIS and at the same time has made a profound influence on the QIS itself [10–16].

Motivated by these facts and our experience [12], we find a novel Hopf algebra as a consequence of the integrability condition, which underlies integrable models with $2 \times 2$ Lax operators and the trigonometric $R$ matrix. This is more general than the well-known quantum Lie algebra and in contrast represents a deformed quadratic algebra (QdA), so-called due to the appearance of generators in quadratic form in the defining algebraic relations. At the same time it unifies a large class of quantum integrable models by generating them in a systematic way through reductions of an ancestor model with explicit Lax operator realization. Note that the Lax operator together with the quantum $R$ matrix defines an integrable system completely, giving also all conserved quantities including the Hamiltonian of the model.

The proposed algebra may be given by the simple relations

$$[S^3, S^{\pm}] = \pm S^{\pm},$$
$$[S^+, S^-] = [M^+ \sin(2\alpha S^3) + M^- \cos(2\alpha S^3)] \frac{1}{\sin\alpha},$$
$$[M^{\pm}, \cdot] = 0, \tag{1}$$

where $M^{\pm}$ are the central elements. We show that (1) is not merely a modification of known $U_q[\mathrm{su}(2)]$, but a QdA underlying an integrable ancestor model, and in effect is dictated by the quantum Yang-Baxter equation (QYBE) [17] $RL\tilde{L} = \tilde{L}LR$. The associated quantum $R(\lambda)$ matrix is the known trigonometric solution related to sine Gordon (SG) [2], while the Lax operator may be taken as

$$L_t^{(\mathrm{anc})}(\xi) = \begin{pmatrix} \xi c_1^+ e^{i\alpha S^3} + \xi^{-1} c_1^- e^{-i\alpha S^3} & 2\sin\alpha S^- \\ 2\sin\alpha S^+ & \xi c_2^+ e^{-i\alpha S^3} + \xi^{-1} c_2^- e^{i\alpha S^3} \end{pmatrix}, \qquad \xi = e^{i\alpha\lambda}, \tag{2}$$

with $c_a^{\pm}$ central to (1) relating $M^{\pm} = \pm\sqrt{\pm 1}\,(c_1^+ c_2^- \pm c_1^- c_2^+)$. The derivation of algebra (1) follows from QYBE by inserting the explicit form (2) and the $R$ matrix and matching different powers of the spectral parameter $\xi$.

Note that (1) is a Hopf algebra [18] and a generalization of $U_q[\mathrm{su}(2)]$. However, unlike Lie algebras or their deformations, due to the presence of multiplicative operators $M^{\pm}$, (1) becomes a quantum deformation of a QdA. Since these operators have arbitrary eigenvalues including zeros, they cannot be removed by scaling and therefore generically (1) is different from the known quantum algebra. Moreover different representations of $M$'s generate new structure constants leading to a rich variety of deformed Lie algebras, which are related to different integrable systems. This fact becomes important for its present application. The appearance of QdA in a basic integrable system should be rather expected, since the QYBE with $R$ matrix having $c$-number elements is itself a QdA. The notion of QdA was introduced first by Sklyanin [19].

The ancestor model can be constructed through representation of (1) in physical variables (with $[u, p] = i$) as [20]

$$S^3 = u, \qquad S^+ = e^{-ip} g(u), \qquad S^- = g(u) e^{ip}. \tag{3}$$

This gives a novel exactly integrable quantum system *generalizing lattice SG model* and associated with the Lax operator (2). It is evident that for Hermitian $g(u)$ only one gets $S^- = (S^+)^{\dagger}$. We show below that through various realizations of the single object $g(u)$ in (3) the ancestor model generates a whole class of integrable models. Their

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Lax operators are derived from (2), while the $R$ matrices are simply inherited. The underlying algebras are given by the corresponding representations of (1).

Evidently, fixing $M^- = 0$, $M^+ = 1$, (1) leads to the well-known quantum algebra $U_q[\mathrm{su}(2)]$ [21]. Now the simplest representation $\vec{S} = \vec{\sigma}$ derives the integrable *XXZ spin chain* [13], while (3) with the corresponding reduction of $g(u)$ [20] yields the *lattice sine-Gordon* model [22] with its Lax operator obtained from (2) with all $c$'s $= 1$.

An asymmetric choice of central elements: $c_{1,2}^+ = 1$, $c_1^- = \frac{1}{c_2^-} = -iq$, $q = e^{i\alpha}$, along with the mapping $S^+ = cA$, $S^- = cA^\dagger$, $S^3 = -N$, $c = (\cot\alpha)^{1/2}$ brings (1) directly to the well-known $q$-oscillator algebra [23,24] and simplifies $g^2(u) = [-2u]_q$ in (3). Therefore using the interbosonic map [25] one gets a bosonic realization for the $q$ oscillator [25]. This realization in turn constructs easily from (2) the Lax operator, which coincides exactly with the discrete version of the quantum *derivative nonlinear Schrödinger equation* (QDNLS) [12]. The QDNLS was shown to be related to the interacting Bose gas with a derivative $\delta$-function potential [26]. Fusing two such models one can further create an integrable *massive Thirring* model described in [2].

Having the freedom of choosing trivial eigenvalues for the central elements, $c_1^- = c_2^+ = 0$ with other $c$'s $= 1$, we obtain another deformed Lie algebra $[S^+, S^-] = \frac{e^{2i\alpha S^3}}{i\sin\alpha}$. This can be realized again by (3) with the related expression for $g(u)$ [20], using which the Lax operator is obtained from (2). The model that results is no other than the *discrete quantum Liouville* model [27]. Note that the present case $M^\pm = \pm\sqrt{\pm 1}$ may be achieved even with $c_1^- \neq 0$, giving the same algebra and hence the same realization. However, the Lax operator which depends explicitly on $c$'s gets changed, reducing (2) to another nontrivial structure. This is an interesting possibility of constructing different useful Lax operators for the same model, in a systematic way. For example, the present construction of the *second Liouville Lax* operator recovers that of [14], invented for its Bethe ansatz solution.

In a similar way the particular case $M^\pm = 0$ can be achieved with different sets of choices: with all $c$'s $= 0$ except (i) $c_a^+ = 1$, (ii) $c_1^\mp = \pm 1$, (iii) $c_1^+ = 1$, all of which lead to the same algebra,

$$[S^+, S^-] = 0, \qquad [S^3, S^\pm] = \pm S^\pm. \tag{4}$$

However, they may generate different Lax operators from (2), which might even correspond to different models, though with the same underlying algebra. In particular, case (i) leads to the *light-cone SG* model, while (ii) and (iii) give two different Lax operators found in [28] and [29] for the same relativistic Toda chain. Since here we get $g(u) = \text{const}$, interchanging $u \to -ip$, $p \to -iu$, (3) yields simply $S^3 = -ip$, $S^\pm = \alpha e^{\mp u}$, generating *discrete time or a relativistic quantum Toda chain.*

Remarkably, all the descendant models listed above have the same trigonometric $R$ matrix inherited from the ancestor model and similarly is true for its rational form, as we will see below. This solves the mystery as to why a wide range of models are found to share the same $R$ matrices. The $L$ operators and the underlying algebras, however, become different, being various reductions of the ancestor Lax operator (2) and the ancestor algebra (1).

We consider now the undeformed $\alpha \to 0$ limit of the proposed algebra (1). It is evident that for the limits to be finite the central elements must also be $\alpha$ dependent. A consistent procedure leads to $S^\pm \to is^\pm$, $M^+ \to -m^+$, $M^- \to -\alpha m^-$, $\xi \to 1 + i\alpha\lambda$, giving the algebraic relations

$$[s^+, s^-] = 2m^+ s^3 + m^-, \qquad [s^3, s^\pm] = \pm s^\pm \tag{5}$$

with $m^+ = c_1^0 c_2^0$ and $m^- = c_1^1 c_2^0 + c_1^0 c_2^1$ as the new central elements. It is again not a Lie but a QdA, since multiplicative operators $m^\pm$ cannot be removed in general due to their allowed zero eigenvalues. Equation (5) exhibits also a noncocommutative feature [18] unusual for an undeformed algebra. Equation (2) at this limit reduces to

$$L_r(\lambda) = \begin{pmatrix} c_1^0(\lambda + s^3) + c_1^1 & s^- \\ s^+ & c_2^0(\lambda - s^3) - c_2^1 \end{pmatrix}, \tag{6}$$

while the $R$ matrix is converted into its rational form, well known for the NLS model [2]. Therefore the integrable systems associated with algebra (5) and generated by the ancestor model (6) would belong to the rational class, all sharing the same rational $R$ matrix.

It is interesting to find that the bosonic representation (3), using the undeformed limit $g_0(u)$ [20] and the interbosonic map [25], reduces into a generalized Holstein-Primakov transformation (HPT)

$$s^3 = s - N, \qquad s^+ = g_0(N)\psi, \qquad s^- = \psi^\dagger g_0(N),$$
$$g_0^2(N) = m^- + m^+(2s - N), \qquad N = \psi^\dagger\psi. \tag{7}$$

It can be checked to be an exact realization of (5), associated with the Lax operator (6). This would serve therefore as an ancestor model of the rational class and represent an integrable *generalized lattice NLS* model.

For the choice $m^+ = 1$, $m^- = 0$, (5) leads clearly to the standard su(2) and for spin $\frac{1}{2}$ representation recovers the *XXX spin chain* [17]. On the other hand, the general form (7) simplies to standard HPT and (6) reproduces the *lattice NLS* model [22].

The complementary choice $m^+ = 0$, $m^- = 1$ reduces (5) to a nonsemisimple algebra and gives $g_0(N) = 1$. This induces a direct realization through oscillator algebra, $s^+ = \psi$, $s^- = \psi^\dagger$, $s^3 = s - N$, and corresponds to another *simple lattice NLS* model [30]. Remarkably, a further trivial choice $m^- = 0$ gives again algebra (4) and therefore the same realization found before for the relativistic case can be used, but now for the *nonrelativistic*

*Toda chain* [2]. The associated Lax operator should however be obtained from (6) along with the rational $R$ matrix.

It should be noted that a bosonic realization of general Lax operators like (2) and (6) can be found also in some earlier works [10,31]. Apart from the discrete models obtained above, one can construct a family of quantum field models starting from their lattice versions. Scaling first the operators such as $p_j$, $u_j$, $c_a^{\pm}$, and $\psi_j$, consistently by lattice spacing $\Delta$ and taking the continuum limit $\Delta \to 0$, one gets $p_j \to p(x)$, $\psi_j \to \psi(x)$, etc. The Lax operator $\mathcal{L}(x, \lambda)$ for the continuum model is then obtained from its discrete counterpart as $L_j(\lambda) \to I + i\Delta\mathcal{L}(x)$. The associated $R$ matrix however remains the same since it does not contain $\Delta$. Thus *integrable field models* like sine Gordon, Liouville, NLS, or the derivative NLS models are obtained from their discrete variants constructed above.

It is possible to further build a new class of models that may be considered as the inhomogeneous versions of the above integrable models. The idea of such construction is to take locally different representations for the central elements, i.e., instead of taking their fixed eigenvalues, one should consider them to be site dependent functions. This simply means that in the expressions of $g(u_j)$ [20] $M^{\pm}$ should be replaced by $M_j^{\pm}$, and consequently in Lax operator (2) all $c$'s should be changed to $c_j$'s. Thus in lattice models the values of central elements may vary arbitrarily at different lattice points $j$ including zeros. This would naturally lead to inhomogeneous lattice models. However since the algebra remains the same they answer to the same quantum $R$ matrices. Physically such inhomogeneities may be interpreted as impurities, varying external fields, incommensuration, etc.

Notice that in the sine-Gordon model, unlike its coupling constant, the mass parameter enters through the Casimir operator of the underlying algebra. Therefore, taking $M_j^+ = -(\Delta m_j)^2$, one can construct a variable mass discrete SG model without spoiling its integrability. In the continuum field limit it would generate a novel *sine-Gordon model with variable mass* $m(x)$ in an external gauge field $\theta(x)$. In the simplest case the Hamiltonian of such a model would be $\mathcal{H} = \int dx \{m(x)(u_t)^2 + m^{-1}(x)(u_x)^2 + 8[m_0 - m(x)\cos(2\alpha u)]\}$. Similar models may arise also in physical situations [32].

An *inhomogeneous lattice NLS* model can be obtained by considering site-dependent values for central elements in (6) and in the generalized HPT (7), where time dependence can also enter as a parameter. As a possible quantum field model it would correspond to equations like *cylindrical NLS* [33] with explicit coordinate dependent coefficients. In a similar way inhomogeneous versions of the Liouville model, relativistic Toda, etc., can be constructed. For example, taking $c_1^a \to c_j^a$ in a nonrelativistic Toda chain we can get a new integrable quantum *Toda chain with inhomogeneity* having the Hamiltonian $H = \sum_j (p_j + \frac{c_j^1}{c_j^0})^2 + \frac{1}{c_j^0 c_{j+1}^0} e^{u_j - u_{j+1}}$.

Another way of constructing inhomogeneous models is to use different realizations of the general QdA (1) or (5) at different lattice sites, depending on the type of $R$ matrix. This may even lead to different underlying algebras and hence different Lax operators at differing sites opening up possibilities of building various exotic inhomogeneous integrable models. Thus, in a simple example of impurity *XXX* spin chain, if we replace its standard Lax operator at a single impurity site $m$ by a compatible $L_{am} = (\lambda + c_m^1)\sigma_a^3$, the Hamiltonian of the model is modified to $H = -(\sum_{j \neq m, m-1} \vec{\sigma}_j \vec{\sigma}_{j+1} + h_{m-1m+1})$, where $h_{m-1m+1} = -(\sigma_{m-1}^+ \sigma_{m+1}^- + \sigma_{m-1}^- \sigma_{m+1}^+) + \sigma_{m-1}^3 \sigma_{m+1}^3$. It gives an integrable quantum spin chain *with a defect*, where the coupling constant has changed sign at the impurity site. If an attempt is made to restore the sign it appears in the boundary condition.

Thus we have prescribed a unifying scheme for quantum integrable systems, where the models can be generated systematically from a single ancestor model with underlying algebra (1). The Lax operators of the descendant models are constructed from (2) or its $q \to 1$ limit (6), while the variety of their concrete representations is obtained from the same general form (3) at different realizations. The corresponding underlying algebraic structures are the allowed reductions of (1). The associated quantum $R$ matrix however remains the same trigonometric or the rational form as inherited from the ancestor model. This fact also reveals a universal character for solving the models through algebraic Bethe ansatz (ABA) [2,34]. The characteristic eigenvalue equation for the ABA is given by $\Lambda_m(\lambda) = \alpha(\lambda) \prod_{j=1}^m f(\lambda_j - \lambda) + \beta(\lambda) \prod_{j=1}^m f(\lambda - \lambda_j)$, where the coefficients $\alpha(\lambda)$ and $\beta(\lambda)$ are the only model-dependent elements; as being eigenvalues of the pseudovacuum they depend on the concrete form of the Lax operator. The main bulk of the expression however is given through functions such as $f(\lambda) = \frac{a(\lambda)}{b(\lambda)}$, i.e., as the ratio of two elements of the $R$ matrix and hence is universal for all models belonging to the same class. Therefore all integrable models solvable through ABA can be given by almost a universal equation based on a general model.

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