Quantized Atom-Field Dynamics in Unstable Cavities

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We develop a quantum description for the dynamics of a single atom inside an unstable optical resonator. For spherical mirrors with a finite Gaussian aperture we find a discrete complete set of normalizable eigenmodes, their biorthogonal adjoint modes, eigenfrequencies, decay rates, and overlap integrals. With these modes we formulate a quantum description for the coupled dynamics of the field and a single atom inside the resonator. We find a strongly geometry and position dependent nonexponential decay probability. Under certain special conditions one gets a modified single-mode description incorporating the Petermann excess noise factor K even on a single-atom level. [S0031-9007(99)09083-3]

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The phenomenon of excess spontaneous emission noise inside unstable resonators was theoretically predicted by Petermann [1] and experimentally observed by measuring the enhanced linewidth of a gain-guided single-mode semiconductor laser [2]. The excess noise factor (K factor) was introduced as the discrepancy between the theoretically expected natural linewidth using the Schawlow-Townes formula [3–5] and the experimentally measured linewidth, which in high gain unstable lasers or in gain-guided semiconductor lasers was observed to reach considerable values [6,7]. A somehow controversial explanation connecting the K factor with the nonorthogonality of the cavity modes was first given by Siegman [8] already nine years ago. For plane wave resonators alternative explanations by a solution of the real space propagation equations were found by several groups [5,9]. In a recent paper Poizat and coworkers [10] pointed out that the effect of excess noise can be mimicked using a simple input-output model involving 3 modes.

Recently the validity of Siegman's rule for a transversely instable geometry has been extensively experimentally tested in a series of beautiful experiments by Woerdman and co-workers [11,12]. Nevertheless, a simple physical picture and a clear mathematical justification to our knowledge is still missing. The central problem is to find a proper quantum description of the electromagnetic field in a finite transversly unstable cavity, as there exists no orthonormal mode basis for the field. What means the existence of "photons/vacuum fluctuations" in such modes? Is it possible to reduce the system to a single effective mode case? Can a spontaneous emission rate be associated with the atomic decay in such a cavity? By reducing the system to the simplest possible nontrivial case and trying to go analytically as far as possible we try to shed some light on these questions.

To build up a consistent physical theory we restrict ourselves to the simplest system containing the essential properties, namely a quasi-1D resonator with length L and two symmetric cylindrical mirrors of focal length f, as depicted in Fig. 1. Surprisingly, even for unstable resonators, i.e., $f < 0$, one still finds self-reproducing field configurations with finite norm, if the mirrors are assumed to have a Gaussian reflectivity profile with width L_G [13].

The slowly varying amplitude of these field "modes" calculated in the paraxial approximation reads within the symmetry plane $z = 0$ as follows:

$$u_n(x) = c_n H_n(p_0 x) e^{-i(k_n/2R_0)x^2} e^{-(x^2/w_0^2)}, \tag{1}$$

with the beam waist $w_0^2 = 2z_0/k_n(1 + r_0^2/z_0^2)$, radius of curvature $R_0 = r_0(1 + z_0^2/r_0^2)$, and transverse scaling $p_0 = \sqrt{ik_n/q_0}$. n is a combined longitudinal and transverse mode index. The only remaining free parameter is the complex source point $q_0 = L/2\sqrt{1 - 4/l} \equiv r_0 + iz_0$, which is directly linked to the cavity parameters; $l = L/f + i/N$ and $N = \pi L_G^2/\lambda L$ would be the Fresnel number of a corresponding hard-edged spherical mirror. With respect to the analytical solvability, we focus on the case of Gaussian apertures instead of the more popular case of hard-edged unstable mirrors which generally lead to much larger K factors.

These quasimodes are calculated by using a self-reproducing condition after one full cavity round trip, i.e., these modes are eigenfunctions of Huygens integral

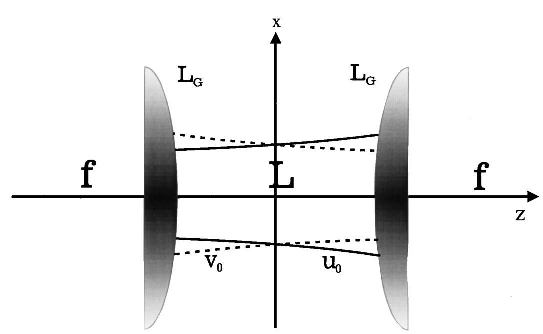

FIG. 1. Scheme of a possible unstable cavity setup within the negative focal length regime.

operator [14] with eigenvalues $\gamma_n = (\frac{q_0 - L/2}{q_0 + L/2})^{2n+1}$. While the modulus of γ_n is one for a stable cavity, its magnitude decreases very rapidly for the unstable case. As the field $a_n = e^{-ik_n z} u_n$ must be multiplied with a real and positive factor for each round trip to ensure the correct boundary conditions on the mirrors we can determine the allowed wave numbers to find $k_{nm} = [m\pi + (n + \frac{1}{2}) \arg\gamma_0]/L$.

As one can show, the modes $u_n(x)$ are complete and biorthogonal to a unique set of adjoint modes

$$v_n(x) = \tilde{c}_n H_n(p_0^* x) e^{i(k_n/2R_0)x^2} e^{-(x^2/w_0^2)}. \tag{2}$$

The normalization factors c_n and $\tilde{c}_n$ are chosen such that

$$\int dx\, u_n^*(x) u_n(x) = 1\,,$$
$$\int dx\, v_n^*(x) u_m(x) = \delta_{nm}\,,$$
$$\int dx\, v_n^*(x) v_n(x) = K_n\,.$$

We thus have found a countable and normable basis set for our cavity field including the geometric losses through finite mirrors. Let us remark here that K_n is simply the Petermann noise factor as defined by Siegman [8] and represents the norm of the adjoint modes. For symmetric cavities it is easy to see that at $z = 0$ the modes are just proportional to the complex conjugate of their adjoint modes, i.e., $v_n(x) = e^{i\varphi_n}\sqrt{K_n}\, u_n^*(x)$ with a given phase φ_n.

It is now interesting to compare the field decay rate

$$\kappa_n = -\frac{c}{2L} \log|\gamma_n| = \frac{c}{L}\left(n + \frac{1}{2}\right) \log\frac{w(L/2)}{w(-L/2)}\,,$$

where we have generalized the z-depending waist function $w^2(z) = 2z_0/k_n[1 + (r_0 + z)^2/z_0^2]$, with the transverse mode spacing modulo π

$$\Delta\omega_\perp = \frac{c}{L} \arg\gamma_0 = \frac{c}{L}[\Psi(L/2) - \Psi(-L/2)]\,, \tag{3}$$

expressed in terms of the generalized Gouy phase $\Psi(z) = \arctan\frac{r_0 + z}{z_0}$, as illustrated in Fig. 2. For unstable cavities ($L/f < 0$ or $L/f > 4$) and a large Gaussian aperture the transverse modespacing modulo π is almost zero, which gives rise to a large frequency degeneracy. Here $\Delta\omega_\perp$ is normally much smaller than κ_0 and the geometric losses are dominant. Note that the edges at the two critical points are washed out for small Fresnel numbers, but the general dependence is only weakly influenced by the Fresnel number. Hence a single transverse mode treatment for unstable resonators is physically very doubtful if not uncorrect. Atoms inside the cavity interact not only with one single-mode/adjoint mode pair, but are substantially coupled to a whole set of spectrally overlapping modes. This will turn out important to calculate the spontaneous emission rate invoking many modes and different K factors.

We will now use the biorthogonal mode set $\{u_n, v_n\}$ to derive a proper quantum description of the transverse field dynamics and apply it to study its interaction with a single atom. The normalization is chosen such that $(u_n, u_n) = 1$, which automatically implies $(v_n, v_n) = K_n$. Note that in the case of symmetric resonators the adjoint modes are proportional to the complex conjugates of the modes, i.e., $v_n(x) = e^{i\varphi_n}\sqrt{K_n}\, u_n^*(x)$. [For stable cavities one has $v_n(x) = u_n(x)$ and $K_n = 1$.] Since these mode pairs fulfill the completeness relation

$$\sum_n v_n^*(x) u_n(x') = \delta(x - x')\,,$$

every field distribution can be expanded uniquely either in the modes or in the adjoint modes. For our purpose, we expand the field operators in the following way:

$$A(x,t) = \sum_n \sqrt{\hbar/2\epsilon_0\omega_n}\,[a_n(t) u_n(x) + b_n^\dagger(t) v_n^*(x)]\,, \tag{4}$$

$$E(x,t) = i\sum_n \sqrt{\hbar\omega_n/2\epsilon_0}\,[a_n(t) u_n(x) - b_n^\dagger(t) v_n^*(x)]\,, \tag{5}$$

which can be inverted to give

$$a_n(t) = -i\sqrt{\epsilon_0/2\hbar\omega_n}\int dx\, v_n^*(x)[E(x,t) + i\omega_n A(x,t)]\,, \tag{6}$$

$$b_n^\dagger(t) = i\sqrt{\epsilon_0/2\hbar\omega_n}\int dx\, u_n(x)[E(x,t) - i\omega_n A(x,t)]\,. \tag{7}$$

$\{a_n, b_n^\dagger\}$ are the annihilation and creation operators for the corresponding mode/adjoint mode pairs, with commutators:

$$[a_n, b_m^\dagger] = \frac{\omega_n + \omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, v_n^*(x) u_m(x) = \delta_{nm}\,, \tag{8}$$

$$[a_n, a_m^\dagger] = \frac{\omega_n + \omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, v_n^*(x) v_m(x) \approx B_{nm}\,, \tag{9}$$

$$[b_n, b_m^\dagger] = \frac{\omega_n + \omega_m}{2\sqrt{\omega_n\omega_m}}\int dx\, u_n^*(x) u_m(x) \approx A_{nm}\,, \tag{10}$$

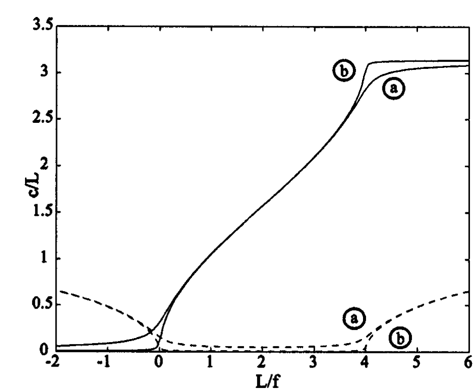

FIG. 2. The loss rate κ_0 (dashed line) and the transverse mode spacing $\Delta\omega_\perp$ (solid line) inside an optical resonator characterized by the ratio L/f and the Fresnel number $(a)N = 5$ and $(b)N = 50$. Note that $\Delta\omega_\perp$ exeeds κ_0 only for stable resonators, where L/f is localized between 0 and 4, but effectively vanishes in loss dominated unstable cases.

where A_{nm} and B_{nm} are overlap integrals between the various cavity modes. Strictly speaking, this field expansion makes sense only for complete frequency degeneracy $\omega_n = \omega_m$, which is approximately valid in the limit of large enough transverse extensions.

Using these operators one can write the Hamiltonian in a canonical form (neglecting in general terms oscillating with $\pm(\omega_n + \omega_m)$, which cancel for symmetric cavities)

$$H = \frac{\epsilon_0}{2} \int dx{:}E^2(x,t) + c^2 B^2(x,t) := \sum_n \hbar\omega_n b_n^\dagger a_n .$$

Note that for unstable systems, where $v_n \neq u_n$ and hence $a_n \neq b_n$ the individual contributions to this Hamiltonian are no longer explicitly Hermitian. However, the non-Hermitian parts cancel approximately within the sum since the overlap matrices in Eqs. (9) and (10) are just inverse, i.e., $\sum_k A_{nk}B_{km} = \sum_k B_{nk}A_{km} = \delta_{nm}$ as a consequence of the completeness relation. This implies a degeneracy between left and right eigenstates given by

$$|n_1, n_2, \ldots) = \frac{b_1^{\dagger n_1}}{\sqrt{n_1!}} \frac{b_2^{\dagger n_2}}{\sqrt{n_2!}} \ldots |0), \tag{11}$$

$$(\widetilde{n_1, n_2, \ldots}| = (0| \ldots \frac{a_2^{n_2}}{\sqrt{n_2!}} \frac{a_1^{n_1}}{\sqrt{n_1!}}, \tag{12}$$

which are biorthogonal fulfilling $(\widetilde{n}|m) = \delta_{nm}$, with $n = \{n_1, n_2, \ldots\}, m = \{m_1, m_2, \ldots\}$. The eigenstates have analogous properties to n-photon Fock states with energy $E_n = \hbar(\omega_1 n_1 + \omega_2 n_2 + \ldots)$. These eigenstates are not orthogonal with respect to the standard scalar product (SP) $(\cdot|\cdot)$. It is, however, possible to introduce a suitably defined SP $\langle\cdot|\cdot\rangle$ so that they are mutually orthogonal. The respective adjoint operation, denoted by $\sim$, has the useful property $\tilde{a}_n = b_n^\dagger$, so that $\langle\cdot|a_n\cdot\rangle = \langle b_n^\dagger \cdot|\cdot\rangle$ with respect to the new SP the left eigenstates are just the adjoint of the right eigenstates, i.e., $(\widetilde{n_1, n_2, \ldots} = \langle n_1, n_2, \ldots|$ and the Hamiltonian reads

$$H = \sum_n \hbar\omega_n \tilde{a}_n a_n . \tag{13}$$

As we are dealing with a lossy system the mode amplitude decays exponentially with a mean rate κ_n. Physically a fraction of the energy is scattered into the continuum modes outside the cavity, which in a proper quantum treatment has to be included by an input-output coupling [15]. After some manipulation one finds the following master equation for the field density operator, i.e.,

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_n \kappa_n\{2a_n\rho\tilde{a}_n - \tilde{a}_n a_n\rho - \rho\tilde{a}_n a_n\}.$$

This master equation yields familiar results for the time evolution of the expectation values of our creation and annihilation operators

$$\dot{\langle a_n\rangle} = (-i\omega_n - \kappa_n)\langle a_n\rangle, \qquad \dot{\langle \tilde{a}_n\rangle} = (i\omega_n - \kappa_n)\langle \tilde{a}_n\rangle,$$

reflecting directly the damped oscillation of the cavity modes and adjoint modes in time. Similarly an n-photon state $\rho(0) = |n_k\rangle\langle n_k|$ decays like

$$\dot{\rho}(0) = -2n\kappa_k\rho(0) + 2n\kappa_k|n - 1_k\rangle\langle n - 1_k| .$$

Finally, let us introduce an atom interacting with the intracavity electromagnetic field. Starting from a minimal coupling Hamiltonian and reducing the interaction to two significant atomic levels with transition frequency ω_A a modified Jaynes-Cummings-Hamiltonian can be found:

$$H_{JC} = H + \frac{\omega_A}{2}\sigma_z - i\sum_n(g_n\sigma_+ a_n - \tilde{g}_n\sigma_-\tilde{a}_n).$$

Formally, everything looks completely familiar except for the coupling, where we have $\tilde{g}_n \neq g_n^*$, or explicitly:

$$g_n = \sqrt{\frac{\hbar\omega_n}{2\epsilon_0}}\, u_n \cdot d_{eg}; \qquad \tilde{g}_n = \sqrt{\frac{\hbar\omega_n}{2\epsilon_0}}\, v_n^* \cdot d_{eg}, \tag{14}$$

with d_{eq} being the atomic dipole matrix element. Again in the special case of stable cavities, we have $v_n = u_n$ and $\tilde{g}_n = g_n$. For symmetric unstable cavities we have $v_n = \sqrt{K_n}e^{i\varphi_n}u_n^*$ and $\tilde{g}_n = \sqrt{K_n}e^{-i\varphi_n}g_n$.

In the following we explore some key consequences of this dynamics. As the most simple nontrivial example we prepare a single excited atom inside the empty cavity and calculate the probability $p(t)$ for a transition to the ground state. This yields

$$p(t) = \sum_n \frac{1}{\hbar^2}|\tilde{g}_n|^2\delta_n(t), \tag{15}$$

$$\delta_n(t) = \frac{1 + e^{-2\kappa_n t} - 2\cos\Delta_n t e^{-\kappa_n t}}{\Delta_n^2 + \kappa_n^2}, \tag{16}$$

where $\Delta_n = \omega_n - \omega_A$ denotes the detuning.

Obviously each contribution is proportional to the corresponding adjoint coupling $|\tilde{g}_n|^2$, because only the atomic lowering terms contribute. If for some conditions this sum is dominated by a particular mode contribution n_0, the transition probability directly involves its K factor. Indeed, for certain parameters (off the critical quasiplanar or quasiconcentric case and moderate detunings) $\delta_n(t)$ is sharply peaked around the ground mode n_0 (cf. Fig. 3), because the damping factor κ_n^2 in the denominator is small only for the lowest order modes. In particular,

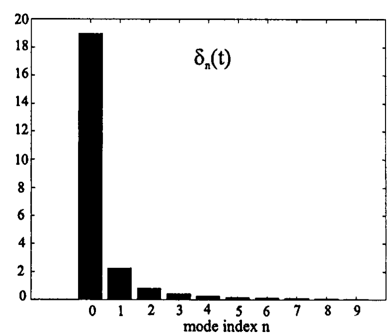

FIG. 3. The factor $\delta_n(t)$ at a fixed time $t = 20\, L/c$ is decreasing quite rapidly with the transverse mode index at least within the unstable regime $\frac{f}{L} = -5$, $N = 100$ near resonance $\Delta_0/\Delta\omega_\perp = 5$.

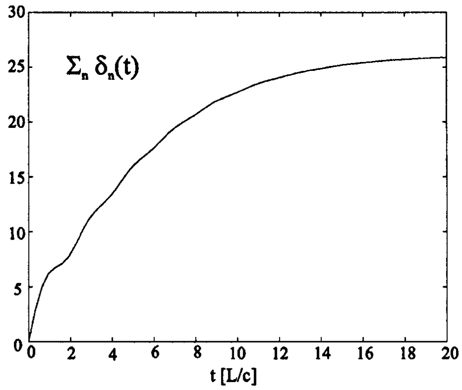

FIG. 4. The transition probability $p(t) \propto \sum_n \delta_n(t)$ for parameters as in Fig. 3 shows an overdamped time dependence $\Delta_0^2/\kappa_0^2 = 0.06$ after growing first approximately linear.

the lowest loss rate κ_0 is already three (or even five, if the atom is localized along the optical axis where odd modes are identically zero) times smaller than for the next participating modes (n_1, n_2). For modes with other longitudinal wave numbers the large detuning Δ_n^2 in the denominator prevents a significant contribution.

Let us now try to extract a spontaneous emission rate from this formula. Approximating the coupling $|\tilde{g}_n|^2$ by a constant over the contributing modes (for large longitudinal mode numbers as for infrared or optical frequencies the modes are varying smoothly with the transverse mode index), we obtain

$$p(t) \approx \frac{1}{\hbar^2} |\tilde{g}_{n_0}|^2 \sum_n \delta_n(t). \tag{17}$$

Assuming moderate values for the excess noise factor, a spontaneous emission rate, which could be inferred from the slope of the transition probability at time $t \approx 0$, reads

$$\gamma = \frac{1}{\hbar^2} |\tilde{g}_{n_0}|^2 \Gamma(\omega_A), \qquad \Gamma(\omega_A) = \frac{\delta}{\delta t} \sum_n \delta_n(t) \Bigg|_{t\approx 0}, \tag{18}$$

where the $\Gamma(\omega_A)$ is determined by the geometric cavity properties. Note that the derivative of each term is zero at time $t = 0$. The notation $\frac{\delta}{\delta t}$ should be understood in the sense that the accumulated transition probability effectively grows linear for small times and the respective slope is to be used here as shown in Fig. 4.

As outlined above for symmetric cavities we have $v_n(x) = e^{i\varphi_n}\sqrt{K_n}u_n^*(x)$ and the spontaneous emission rate within these approximations reads

$$\gamma = \Gamma(\omega_A) \frac{\omega_A}{2\hbar\epsilon_0} |u_{n0} \cdot d_{eg}|^2 K_{n_0}. \tag{19}$$

Indeed it is enhanced by the excess noise factor K_{n_0} of the lowest order transverse mode n_0 with the smallest detuning. The validity of all these last approximations is, as one can see, rather limited, which renders the excess noise factor K a rather superficial quantity in this context. One can only expect it to give some qualitative estimates. However, in the case of an active medium, as, e.g., a laser, where gain compensates the losses for a particular selected mode, the situation can be different and Siegman's predictions should be also quantitatively accurate.

In conclusion, we have shown that a quantum description of the field in an unstable cavity in terms of quasimodes implies important modifications compared to standard cavity QED. Although we have investigated only a small fraction of its implications, we have found new and somehow surprising dynamical consequences. We believe to have shown that under certain well-defined conditions as mentioned in our work the concept of the K factor does have a meaning even for a single atom. This has created some controversy in the past and we think that our work shows the applicability as well as the limitations of this kind of physical picture. In general, many K factors will show up in the dynamics and only one gets some effective average effect. This work was supported by the Austrian FWF under Grants No. S6506 and No. S13435.

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