Quantum Hall Plateau Transitions in Disordered Superconductors

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We study a delocalization transition for noninteracting quasiparticles moving in two dimensions, which belongs to a new symmetry class. This symmetry class can be realized in a dirty, gapless superconductor in which time-reversal symmetry for orbital motion is broken, but spin-rotation symmetry is intact. We find a direct transition between two insulating phases with quantized Hall conductances of zero and two for the conserved quasiparticles. The energy of the quasiparticles acts as a relevant symmetry-breaking field at the critical point, which splits the direct transition into two conventional plateau transitions. [S0031-9007(99)09003-1]

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The variety of universality classes possible in single-particle models of disordered conductors is now appreciated to be quite rich. Three of these classes were identified early in the development of weak localization theory [1]: they are distinguished by the behavior of the system under time reversal, by its spin properties, and are termed orthogonal, unitary, and symplectic, in analogy with Dyson's classification of random matrice ensembles. Further alternatives can arise by two different mechanisms. First, in certain contexts, most notably the integer quantum Hall effect, the nonlinear σ model describing a two-dimensional system may admit a topological term [2], which results in the existence of extended states at isolated energies in an otherwise localized spectrum. Physically, such systems have more than one distinct insulating phase, each characterized by its number of edge states and separated from other phases by delocalization transitions. Second, it may happen that the Hamiltonian has an additional, discrete symmetry, absent from Dyson's classification. This is the case in two-sublattice models for localization, if the Hamiltonian has no matrix elements connecting states that belong to the same sublattice [3,4]. It is also true of the Bogoliubov–de Gennes formalism for quasiparticles in a superconductor with disorder [5,6]. One consequence of this extra symmetry is that, at a delocalization transition, critical behavior can appear not only in two-particle properties such as the conductivity, but also in single-particle quantities, such as the density of states.

Universality classes in systems with extra discrete symmetries of this kind have attracted considerable attention from various directions. A general classification, systematizing earlier discussions [3,5], has been set out by Altland and Zirnbauer [6], who examined mesoscopic normal-superconducting systems as zero-dimensional realizations of some examples. Very recently, quasiparticle transport and weak localization have been studied in disordered, gapless superconductors in higher dimensions, with applications to normal-metal/superconductor junctions and to thermal and spin conductivity in high temperature superconductors [7–10]. Separately, the behavior of massless Dirac fermions in two space dimensions, scattered by particle-hole symmetric disorder in the form of a random vector potential, has been investigated intensively [11] as a tractable example of a disordered critical point. And much before this, the one-dimensional tight binding model with random nearest-neighbor hopping was shown [12] to have a delocalization transition and divergent density of states at the band center, the energy invariant under the sublattice symmetry.

In this paper, we study a new delocalization transition in two dimensions that combines both of the above features: the transition separates phases with different quantized Hall conductances for the quasiparticles, and it occurs in a system which has a discrete microscopic symmetry. This transition can take place in a gapless superconductor under appropriate conditions: time-reversal invariance for orbital motion must be broken by an applied magnetic field, but the Zeeman coupling should be negligible, so that the full spin-rotation invariance remains intact. A candidate system is a granular superconducing film in a magnetic field which frustrates the Jospehson coupling between the grains, so that the order parameter is spatially random [6]. Another is a dirty superconductor in which the order parameter has $d_{x^2-y^2} + id_{xy}$ symmetry [13]. Formally, we suggest that the model whose behavior we examine numerically is a representative of the symmetry class labeled C by Altland and Zirnbauer [6], and that the delocalization transition is associated with a topological term allowed in the field theory, as noted by Senthil *et al.* [9]. The possibility of quantum Hall states in superconductors with broken parity and time-reversal symmetry has been emphasized by Laughlin [13]. A direct transition into such a phase, in the presence of disorder, is of particular interest in connection with the theory of the quantum Hall effect, since it is between phases with Hall conductance differing by two units. Changes of Hall conductance by more than one unit at

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a delocalization transition are precluded in generic systems, by the standard scaling flow diagram [14] for the integer quantum Hall effect, and are possible in the model we study only because of additional symmetry. In the presence of a symmetry-breaking coupling in the model, of strength Δ, the transition occurs in two stages with a separation that varies as Δ^{ϕ} for small Δ, where $\phi \approx 1.3$. Such a coupling is introduced if one treats quasiparticle motion at finite energy.

The system we consider is formulated as a generalization of the network model [15] for the quantum Hall plateau transition. The original version of this model describes guiding center motion of spin-polarized electrons within one Landau level of a disordered, two-dimensional system in a magnetic field. It therefore has broken time-reversal symmetry for orbital motion and contains no spin degree of freedom. It is specified in terms of scattering or transfer matrices, defined on links and nodes of a lattice. The model can be generalized in various ways. Spin can be incorporated by allowing two amplitudes to propagate on each link. This has been done previously [16–18], with the intention of describing a spin-degenerate Landau level in which the two spin states are coupled by spin-orbit scattering. In that case, the random U(1) phases which characterize propagation on the links of the original model are replaced with random U(2) matrices, mixing the two spin states without any rotational symmetry. In the work presented here, we choose instead random SU(2) matrices, preserving spin-rotational symmetry.

In detail, the transfer matrix associated with each link of the model is an SU(2) spin-rotation matrix of the form

$$\mathbf{U} = \begin{pmatrix} e^{i\delta_1}\sqrt{1-x}, & -e^{i\delta_2}\sqrt{x}, \\ e^{-i\delta_2}\sqrt{x}, & e^{-i\delta_1}\sqrt{1-x} \end{pmatrix}, \qquad (1)$$

where δ_1, δ_2, x are random. The transfer matrix at the nodes is parametrized by $\epsilon \pm \frac{1}{2}\Delta$ so that the transmission probability for the two spin states is $\{1 + \exp[-\pi(\epsilon \pm \frac{1}{2}\Delta)]\}^{-1}$, respectively. The value of ϵ determines the Hall conductance of the system, as measured at short distances: varying ϵ drives the model through the delocalization transition. A nonzero value for Δ breaks spin-rotation invariance, and will, in fact, change the universality class for the transition. Collecting factors, the transfer matrix across one node and the links connected to it is a 4×4 matrix of the form [16]

$$\mathbf{T} = \begin{pmatrix} \mathbf{U_1} & 0 \\ 0 & \mathbf{U_2} \end{pmatrix}\begin{pmatrix} \mathbf{C} & \mathbf{S} \\ \mathbf{S} & \mathbf{C} \end{pmatrix}\begin{pmatrix} \mathbf{U_3} & 0 \\ 0 & \mathbf{U_4} \end{pmatrix}, \qquad (2)$$

where $\mathbf{S} = \mathrm{diag}\,(\alpha_-, \alpha_+)$, $\mathbf{C}^2 - \mathbf{S}^2 = \mathbf{1}$, $\alpha_\pm = \exp[-\pi(\epsilon/2 \pm \Delta/4)]$, and $\mathbf{U}_i$ are as given in Eq. (1). From these 4×4 transfer matrices, $\mathbf{T}$, one can build up a larger transfer matrix, of size $2M_l \times 2M_l$, with M_l even, to describe scattering in one slice of a system of width M_l links (which has $M \equiv 2M_l$ scattering channels) by using independent realizations of $\mathbf{T}$ as diagonal blocks of the larger matrix.

Both the 4×4 transfer matrix, $\mathbf{T}$, and the larger ones derived from it, which we denote here also by $\mathbf{T}$, are invariant under an antiunitary symmetry operation representing spin reversal. The corresponding operator is $\mathbf{Q} = \mathbb{1} \otimes i\tau_y \cdot K$, where the Pauli matrix τ_y acts on the two spin states propagating along each link, $\mathbb{1}$ is the $M_l \times M_l$ unit matrix, and K is complex conjugation. The symmetry, $\mathbf{QTQ}^{-1} = \mathbf{T}$, holds for $\Delta = 0$ only; it implies that the Lyapunov exponents of the transfer matrix product have a twofold degeneracy at $\Delta = 0$, which we exploit in the analysis of our simulations, as described below.

It is possible to relate this network model to a Hamiltonian, H, following Ref. [19], by constructing a unitary matrix which can be interpreted as the evolution operator for the system, for a unit time step. Taking the continuum limit, one obtains, in the case of the original network model for a spin-polarized Landau level, a two-component Dirac Hamiltonian with random mass, scalar potential, and vector potential. For the network model of current interest, we get instead a four-component Dirac Hamiltonian of the form

$$H = (\sigma_x p_x + \sigma_z p_z + m\sigma_y) \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\alpha} \cdot \boldsymbol{\tau}, \qquad (3)$$

where σ_i and τ_i for $i = x, y, z$ are two copies of the Pauli matrices, $\mathbb{1}$ is the 2×2 unit matrix, p_x and p_z are the two components of the momentum operator in the plane of the system, the mass m is proportional to ϵ, the distance from the critical point, and the real, three-component vector, $\boldsymbol{\alpha}$, is a random function of position. This Hamiltonian has the symmetry $\mathbf{Q}H\mathbf{Q}^{-1} = -H$ [20], which is the defining feature of the class labeled C by Altland and Zirnbauer [6]. A nonzero value for Δ in the network model introduces an additional term, $H' = \Delta\sigma_y \otimes \tau_z$ into the Dirac Hamiltonian, breaking the symmetry. Equally, since the symmetry relates eigenstates with energies $\pm E$, and leaves invariant only those at energy $E = 0$, nonzero E, like Δ, acts as a symmetry-breaking perturbation.

The models represented by Eqs. (2) and (3) describe propagation of quasiparticles which are conserved, and which are obtained within the Bogoluibov–de Gennes formalism by making a particle-hole transformation on states with one spin orientation (see, for example, Ref. [6]). Specifically, starting from the Bogoluibov–de Gennes Hamiltonian for a singlet superconductor,

$$H_S = \sum_{ij} [h_{ij}(c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}) + \Delta_{ij} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger + \Delta_{ij}^* c_{j\downarrow} c_{i\uparrow}]$$

and introducing transformed operators, $\gamma_{i\uparrow} = c_{i\uparrow}$ and $\gamma_{i\downarrow}^\dagger = c_{i\downarrow}$, one has

$$H_S = \sum_{ij} (\gamma_{i\uparrow}^\dagger \quad \gamma_{i\downarrow}^\dagger) \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{i,j}^* & -h_{ij}^T \end{pmatrix} \begin{pmatrix} \gamma_{j\uparrow} \\ \gamma_{j\downarrow} \end{pmatrix}. \qquad (4)$$

This Hamiltonian, like the SU(2) network model that we simulate, has the symmetry $\mathbf{Q}H_S\mathbf{Q}^{-1} = -H_S$ [6], where here $\mathbf{Q} = i\tau_y K$ and τ_y acts on the particle-hole spinor of

Eq. (4). For singlet pairing (maintaining spin-rotation invariance) Δ_{ij} is symmetric while $h_{i,j}$ is Hermitian and the symmetry under $\mathbf{Q}$ is obvious; time-reversal symmetry is broken if $h \neq h^*$ or $\Delta \neq \Delta^*$. Since quasiparticle charge in a superconductor is not conserved, and charge response is controlled by the condensate, the localization problem of interest in a disordered superconductor, as emphasized in Ref. [9], involves spin and energy transport, rather than charge transport. We stress that the Hall conductance examined below is a property of quasiparticles described by a Hamiltonian such as Eq. (3). We emphasize also that since our identification of the SU(2) network model as a description of a superconductor is based on symmetry arguments rather than a microscopic mapping, we expect only to determine universal aspects of the plateau transition from our calculations.

We study the model defined from Eq. (2) at a range of values for ϵ and Δ. Preliminary calculations, reported earlier [21], were limited to $\Delta = 0$. We compute the normalized localization length, ξ_M/M_l, for strips of width $M = 16, 32, 64$, with periodic boundary conditions. For small Δ we find it necessary also to use $M = 128, 256$, in order to identify more clearly the critical properties. The matrices $\mathbf{U}$ associated with each link are distributed with the Haar measure on SU(2). Runs were carried out for strips of length 60 000 (for $M = 16, 32$), 240 000 (for $M = 64$), and 480 000 ($M = 128, 256$). The errors are typically 0.5%, except for $M = 256$.

The behavior at nonzero Δ is shown in Fig. 1. For $\Delta = 2.0$, extended states (ξ_M/M_l independent of M) appear clearly at two energies, $\pm\epsilon_c$, with $\epsilon_c(\Delta = 2) \approx 0.6$. A one-parameter scaling fit for $\xi_M/M = f[(\epsilon - \epsilon_c)M^{1/\nu_0}]$ yields $\nu_0 \approx 2.5$, the conventional quantum Hall exponent [16,21,22]. For $\Delta = 0.2$, ϵ_c is too small to be resolved by this method (Fig. 1, region II). Proceeding in this way at a range of values for Δ, we construct the phase diagram for the model shown in Fig. 2. With $\Delta \neq 0$, the two phases at $\epsilon \ll -1$ and $\epsilon \gg +1$ are separated by an intermediate, small ϵ, phase. Counting edge states in each phase in the strong-coupling limit ($\epsilon \to \pm\infty$ at fixed Δ, and $\Delta \to \infty$ at $\epsilon = 0$, respectively), we find that the quasiparticle Hall conductance takes the values 0, 1, and 2 in successive phases with increasing ϵ. If Δ is made smaller, the boundaries of the intermediate phase approach each other, and at $\Delta = 0$ there is a direct transition between phases with Hall conductance differing by two units.

To study this direct transition, we examine behavior at small Δ in more detail. On the line $\Delta = 0$, the localization length diverges at a single critical point, $\epsilon_c = 0$, with an exponent, $\nu = 1.12$, which is different from that at the conventional plateau transition (Fig. 3, curve I). Close to this line, scaling with system size is quite complex and, in particular, the variation of ξ_M/M with M is not monotonic. In order to extract scaling properties at small, nonzero Δ, we monitor the

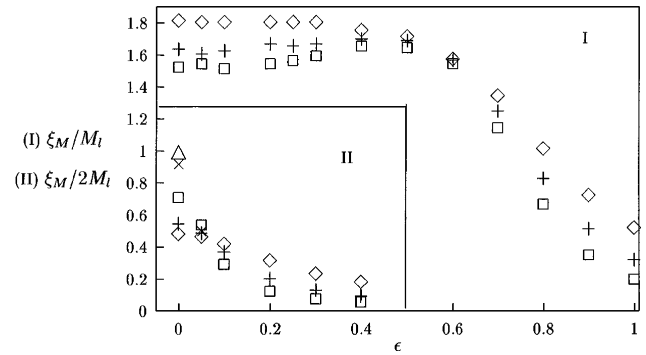

FIG. 1. Normalized localization length ξ_M/M_l. The symbols $\diamond$, $+$, $\square$, $\times$, and $\triangle$ correspond to system widths M of 16, 32, 64, 128, and 256, respectively. The symmetry-breaking parameter is $\Delta = 2$ (region I) or $\Delta = 0.2$ (region II).

deviation from Kramer's degeneracy of the smallest two positive Lyapunov exponents, λ_1 and λ_2, of the transfer matrix product that represents the sample, defining $\bar{\xi} = M(\lambda_2 - \lambda_1)$. Finite size scaling of both $\bar{\xi}$ and ξ_M is shown in Fig. 3 (curves II and III), as a function of Δ along the symmetry line $\epsilon = 0$. We find for deviations from Kramers degeneracy $\bar{\xi} = f_1(\Delta M_l^{1/\mu})$ and for the localization length $\xi_M/M = f_2(\Delta M_l^{1/\mu})$, with $\mu \approx 1.45$.

We propose, then, that $\epsilon = \Delta = 0$ is a critical point at which ϵ parameterizes the symmetry-preserving relevant direction, and Δ is a symmetry-breaking field, so that ξ_M/M_l is described near the fixed point by a two-parameter scaling function,

$$\xi_M/M_l = f(\epsilon M_l^{1/\nu}, \Delta M_l^{1/\mu}), \tag{5}$$

with $\nu = 1.12$ and $\mu \approx 1.45$. In the presence of a symmetry-breaking field, $\Delta \neq 0$, scaling flow is away from the new fixed point, giving quantum Hall plateau phases except on trajectories which connect this unstable fixed point to fixed points at finite Δ, representing the conventional universality class for plateau transitions. At these, a finite critical $\epsilon_c(\Delta)$ is expected with an exponent ν_0, so that $\xi \sim [\epsilon - \epsilon_c(\Delta)]^{-\nu_0}$. Since the values of $\epsilon M^{1/\nu}$ and $\Delta M^{1/\mu}$ on a critical trajectory serve to define a one-parameter curve, we expect $\epsilon_c(\Delta) = \pm c\Delta^{\mu/\nu}$. Thus,

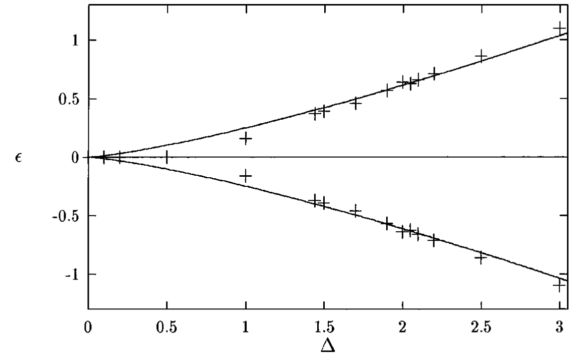

FIG. 2. Phase diagram. The $+$ symbols indicate fitted positions of extended states, and the lines are $\epsilon = \pm 0.25\Delta^{1.3}$.

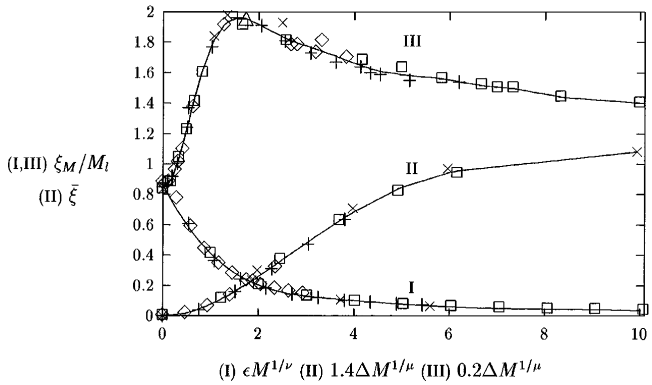

FIG. 3. Scaling functions: (I) Normalized localization length ξ_M/M_l as a function of $\epsilon M^{1/\nu}$ with $\nu = 1.12$ for $\Delta = 0$. (II) Deviation from Kramer's degeneracy $\bar{\xi}$ as a function of $\Delta M^{1/\mu}$ with $\mu = 1.45$ for $\epsilon = 0$. (III) ξ_M/M_l as a function of $\Delta M^{1/\mu}$ with $\mu = 1.45$ for $\epsilon = 0$. Symbols denote system widths as in Fig. 1.

as Δ approaches zero, extended states coalesce, having a separation, $2\epsilon_c \propto \Delta^{1.3}$ (the line in Fig. 2), which is much smaller than Δ, their separation in the absence of coupling between the two spin orientations.

A further aspect of the critical point which is of interest, but not accessible within our numerical approach, is the behavior of single-particle quantities such as the density of states, discussed recently in Ref. [23]. We expect for the Hamiltonian of Eq. (3) a finite density of states at all energies provided $\Delta \neq 0$, and singularities in the density of states at zero energy when $\Delta = 0$, with a different nature according to whether $\epsilon = 0$ or $\epsilon \neq 0$.

In conclusion, we have shown that quantum Hall plateau transitions belonging to a new universality class occur in a model for a gapless superconductor which is invariant under spin rotations, but which has time-reversal symmetry broken for orbital motion. In contrast to the conventional plateau transition, the Hall conductance for conserved quasiparticles changes at this transition by two units. We have examined critical behavior, and shown that there is a symmetry-breaking perturbation which is relevant at the critical point, splitting the transition into two, with extended states that coalesce as the symmetry-breaking field is removed.

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Note added.—Since submission of this paper, two replated preprints have appeared. One, Ref. [23], includes a derivation of the SU(2) network model from a theory of edge states for a $d_{x^2-y^2} + id_{xy}$ superconductor. In the other, Ref. [24], the SU(2) network model is mapped onto the two-dimensional percolation problem.

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