Power Laws in Nonlinear Granular Chain under Gravity

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The signal generated by a weak impulse propagates dispersively in a gravitationally compacted granular chain. For a power-law-type contact force, we show analytically that the type of dispersion follows power laws in depth. The power law for a grain displacement signal is given by $h^{-(1/4)[1-(1/p)]}$ where h and p denote depth and the exponent of the contact force, respectively, and the power law for the grain velocity is $h^{-(1/4)[(1/3)+(1/p)]}$. Other depth-dependent power laws for oscillation frequency, wavelength, and period are given by combining the above two expressions and the phase velocity power law $h^{(1/2)[1-(1/p)]}$. We verify these power laws by comparing with the data obtained by numerical simulations. [S0031-9007(99)08953-X]

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Physics of granular materials has attracted great interest recently [1], since these materials are ubiquitous around us and their properties are unique and also useful in many applications [2,3]. The propagation of a sound or a weak elastic wave in a granular medium is also one of the interesting subjects related to the properties of granular matter [4]. A rather simple system, the granular chain with Hertzian contact [5], has been revived by finding a soliton in transmitting elastic impulse. This soliton, existing in a highly nonlinear regime of a horizontal Hertzian chain was first predicted by Nesterenko [6] and its experimental verification was performed by Lazaridi and Nesterenko [7] and recently by Coste *et al.* [8]. Even though three-dimensional granular systems may not follow simple Hertzian contact force law due to geometrical effect [9], the one-dimensional granular chain with nonlinear contact force is still interesting [10]. It may describe a fundamental feature of the dynamics of nonlinear granular chain which appears in many areas of nature. In addition, the one-dimensional system is usually the starting point of studying higher-dimensional systems.

It is well known that the velocity of an elastic impulse scales as $P^{1/6}$ or $h^{1/6}$ for the Hertzian chain [9], where P is the pressure, linearly proportional to the depth h for vertical chain. Sinkovits and Sen [11] extended this to arbitrary nonlinear contact force of power-law type $F \propto \delta^p$, where δ denotes overlapped distance between adjacent grains. They showed that the signal velocity v_{ph} scales as $h^{[1-(1/p)](1/2)}$ for $p \geq 1$ at large h. This has been simply obtained by considering the well-known relation $v_{\mathrm{ph}} \propto \sqrt{\mu}$, where μ is the elastic constant which is given by $\mu \sim h^{1-p}$ for the above power-law-type contact force. As far as we know, however, no power-law dependences on depth of the signal characteristics, such as oscillation frequency, period, and wavelength have been found in the gravitationally compacted chain.

In this work, we study the propagation of acoustic or weak impulses in the gravitationally compacted granular chain. We derive analytically the power-law behaviors of signal characteristics which depend on depth or time. We treat here a rather weak impulse which makes grain motion oscillatory and can be treated analytically even though it contains nonlinearity. The other extreme which is a highly nonlinear regime has been studied by Nesterenko [6]. Initial impulse may be used as a parameter which controls the solitariness of the signal. The power-law behaviors for a wide range of impulse will be discussed in a separate work [12].

We would like to obtain analytically the exponents of various power laws, such as grain displacement, grain velocity, and oscillation period, frequency, and wavelength. We first solve the equation of motion of a grain displacement under gravity in the small oscillation or weak impulse regime in which the equation of motion under gravity can be mapped into the equation for the horizontal linear chain with varying force constant at each contact. The normal mode solution of the equation of motion can be obtained analytically in the continuum or long wavelength limit. The asymptotic behavior of the normal mode gives rise to the correct power-law behavior in depth, since the equation of motion has been changed into a linear form. Once we get the information on the grain velocity, all sorts of power laws mentioned above can be obtained. Since the equation of motion for grain velocity is not linear, the normal mode solution may not work to obtain power law in depth. Therefore, we construct fully dispersive forms describing displacement and velocity signal and obtain their depth-dependence behaviors. Our solution is quite general and gives rise to generic power laws for arbitrary exponent p of the contact force in the oscillating regime.

The equation of motion of nth grain at z_n is given by

$$m\ddot{z}_n = \eta[\{\Delta_0 - (z_n - z_{n-1})\}^p - \{\Delta_0 - (z_{n+1} - z_n)\}^p] + mg\,, \qquad (1)$$

where z_n is the distance from the top of the chain to the center of the nth spherical grain, m is the mass of grain, Δ_0 is the distance between adjacent centers

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of the spherical grain, and η is the elastic constant of grain. Therefore, the overlap between adjacent grains at nth contact is $\delta_n = \Delta_0 - (z_{n+1} - z_n)$. It is usually impossible to solve general nonlinear problems in an analytical way. Therefore we may not solve the nonlinear differential equation of Eq. (1) exactly. But we may treat it analytically in a small oscillation regime which can be achieved by applying a weak impulse.

For this purpose, we introduce a new variable

$$\psi_n = z_n - n\Delta_0 + \sum_{l=1}^{n} \left(\frac{mgl}{\eta}\right)^{1/p}, \tag{2}$$

where the last term is the sum of overlaps up to the nth grain and set $z_0 = \psi_0 = 0$. This change of variable makes Eq. (1) into an equation for the linearized horizontal chain with varying force constant at each contact, i.e.,

$$m \frac{\partial^2}{\partial t^2} \psi_n = -\mu_n(\psi_n - \psi_{n-1}) + \mu_{n+1}(\psi_{n+1} - \psi_n), \tag{3}$$

where $\mu_n = \mu_1 n^{[1-(1/p)]}$ is the force constant of the nth contact and $\mu_1 = mpg(\frac{\eta}{mg})^{1/p}$ is the force constant of the first contact. Use of the condition of small oscillation

$$|\psi_n - \psi_{n-1}| \ll \left(\frac{mgn}{\eta}\right)^{1/p} \tag{4}$$

has been made to obtain Eq. (3) approximately. The expression of Eq. (3) in the continuum limit, i.e., the lattice constant $a = \delta h \to 0$, is given by

$$\frac{\rho}{\tau_1} \frac{\partial^2}{\partial t^2} \psi(h,t) = a^{[(1/p)-1]} \frac{\partial}{\partial h}\left[\mu(h) \frac{\partial}{\partial h} \psi(h,t)\right], \tag{5}$$

where $\mu(h) = h^{[1-(1/p)]}$ denotes the depth dependence of force constant, and $\rho = m/a$ and $\tau_1 = \mu_1 a$ are the linear density and the tension of a chain at the first contact, respectively. We set $c_1 = \sqrt{\tau_1/\rho}$ which is the well-known speed of wave in the string of tension τ_1 and line density ρ.

We now choose $\psi_\zeta(h,t) = u_\zeta(h) e^{-ic_1 a^{(1/2p)-(1/2)}\zeta t}$ as a normal mode solution. Then the depth-dependent function $u_\zeta(h)$ satisfies

$$\frac{d^2}{dh^2} u_\zeta(h) + \frac{1-(1/p)}{h} \frac{d}{dh} u_\zeta(h) + \frac{\zeta^2}{h^{1-(1/p)}} u_\zeta(h) = 0, \tag{6}$$

which is a type of Bessel's differential equation [13]. If we consider a solution propagating to the positive h direction, the solution of Eq. (6) is given by the Hankel function [13]

$$u_\zeta(h) = h^\xi H_\nu^{(1)}(\theta h^\gamma), \tag{7}$$

where $\xi = \frac{1}{2p}, \gamma = \frac{1}{2} + \xi = \frac{1}{2}(1 + \frac{1}{p}), \theta = \frac{\zeta}{\gamma}, \nu = \frac{\xi}{\gamma} = \frac{1}{1+p}$.

The asymptotic form of Eq. (7) at large h for a fixed ν is

$$u_\zeta(h) \approx \sqrt{\frac{2}{\pi\theta}}\, h^{\xi-(\gamma/2)} e^{i[\theta h^\gamma - (\pi/2)\nu - (\pi/4)]} \tag{8}$$

and the displacement function becomes

$$\psi_\zeta(h,t) \approx h^{\xi-(\gamma/2)} e^{i[(\zeta/\gamma)h^\gamma - c_1 a^{(1/2p)-(1/2)}\zeta t]}. \tag{9}$$

The amplitude of displacement signal is given by the envelope function of the asymptotic solution, which scales as

$$A(h) \propto h^{\xi-(\gamma/2)} = h^{-(1/4)[1-(1/p)]}. \tag{10}$$

The phase velocity $v_{\rm ph}$ is obtained by setting the phase of Eq. (9) constant, i.e., $\frac{\zeta}{\gamma} h^\gamma - c_1 a^{(1/2p)-(1/2)}\zeta t =$ constant. We obtain

$$v_{\rm ph} = \frac{dh}{dt} = c_1 a^{(1/2)[(1/p)-1]} h^{(1/2)[1-(1/p)]}. \tag{11}$$

The group velocity scales as that of phase velocity and the dispersion relation is linear but depth dependent in this case.

Since Eq. (5) is a linear differential equation, the envelope function of a normal mode solution in Eq. (9) can describe the depth dependence of the displacement signal which may be given by the linear combination of all normal modes of different frequency. Therefore, one can expect that the depth-dependent behavior predicted by Eq. (10) may agree well with the data given by numerical simulation which will be shown in what follows.

The normal mode solution of Eq. (9) alone, however, cannot give appropriate predictions on the changes in frequency, wavelength, and period of the signal as it propagates down. The information on the grain velocity may give the characteristics of signal dispersion by combining it with the displacement and velocity of signal. To obtain the depth dependence of grain velocity we write Eq. (5) as

$$\frac{\partial}{\partial t} v(h,t) = \frac{\partial}{\partial h}\left[\mu(h) \frac{\partial}{\partial h} \psi(h,t)\right]. \tag{12}$$

We set the constant factor of Eq. (5) unity for our convenience, since it has nothing to do with depth dependence behavior. To draw the depth-dependent behaviors of grain velocity, frequency, etc., we set the frequency $\omega(h) \propto h^\alpha$, the displacement function

$$\psi(h,t) \propto h^{-(1/4)[1-(1/p)]} e^{ik(h)h - i\omega(h)t}, \tag{13}$$

and the grain velocity

$$v(h,t) \propto h^\beta e^{ik(h)h - i\omega(h)t + \phi}, \tag{14}$$

where α and β will be determined and ϕ is the phase difference between displacement and velocity signal. The depth-dependent wave number $k(h)$ is given by $k(h) = v_{\rm ph}(h)/\omega(h)$.

Since we have two unknowns α and β to be determined, we need two independent equations for these. One is given by Eq. (12) and the other the relation $\omega(h) \propto v(h)/\psi(h)$. Substituting Eqs. (13) and (14) into

Eq. (12) gives rise to $\beta = \frac{1}{4} - \frac{5}{4p} - 2\alpha$ and the relation $\omega(h) \propto v(h)/\psi(h)$ yields $\alpha = \beta + \frac{1}{4} - \frac{1}{4p}$. We obtain the power-law behaviors of grain velocity and frequency from these two equations as follows:

$$v(h) \propto h^{-(1/4)[(1/3)+(1/p)]}, \tag{15}$$

$$\omega(h) \propto h^{[(1/6)-(1/2p)]}. \tag{16}$$

The characteristic time of oscillation which is expressed by the period is given by the inverse of frequency or the ratio of displacement to grain velocity, i.e.,

$$T(h) = \frac{A(h)}{v(h)} \propto \omega(h)^{-1} \propto h^{-[(1/6)+(1/2p)]}. \tag{17}$$

The characteristic length of oscillation, on the other hand, which is expressed by wavelength is given by multiplying $T(h)$ by phase velocity, i.e.,

$$\lambda(h) = T(h)v_{\mathrm{ph}}(h) \propto h^{1/3}. \tag{18}$$

Interestingly enough, characteristic length of oscillation does not depend on contact force within the linear approximation of Eq. (3).

We now compare the above results with molecular dynamics simulations performed for Eq. (1) for arbitrary p. To perform numerical simulation for Eq. (1), we choose a vertical chain of $N = 2 \times 10^3$ grains and neglect plastic deformation. As a calculational tool, we use the third-order Gear predictor-corrector algorithm [14]. We choose 10^{-5} m, 2.36×10^{-5} kg, and 1.0102×10^{-3} s as the units of distance, mass, and time, respectively. These units gives the gravitational acceleration $g = 1$ [10]. We set the grain diameter 100, mass 1, and the elastic constant η of Eq. (1) is given by $\eta = (1 + p)b$, where b depending on modulus is chosen as 5657 for this molecular dynamics simulation. The equilibrium condition

$$mgn = \eta\delta_n^p \tag{19}$$

has been used for the $(n + 1)$th grain of a vertical chain. Even though there is a criterion [8,15] for initial impulse to neglect plastic deformation and viscoelastic dissipation in experimental situations, we do not care about that criterion for the numerical simulation. For the purpose of this work, however, we choose a rather weak initial impulse $v_i = 0.1$ in our program units. There is a regime of initial impulse in which the signal follows the same power laws. This will be shown in a separate paper [12].

Figure 1 shows the snap shots of amplitudes (a) and corresponding grain velocity signals (b) propagating down the vertical chain with Hertzian contact ($p = 3/2$) [10]. The leading amplitude of displacement of each signal in Fig. 1(a) corresponds to the leading part of each velocity signal in Fig. 1(b).

We focus on the leading amplitudes of displacement and velocity signal and plot them in $\log_{10}$-$\log_{10}$ scale in Fig. 2 which shows that both displacement and velocity peak decrease in power laws of depth. The explicit expressions for the depth-dependent behaviors of leading amplitudes of displacement and velocity are given by $A_{\max}(h) \propto h^{-0.0835\pm0.0003}$ and $v_{\max}(h) \propto h^{-0.2500\pm0.0001}$. We also obtain other depth-dependent power laws showing dispersiveness of the signal. They are the elasped time to reach to maximum amplitude, $T_{\max}(h)$, which describes the period and the number of

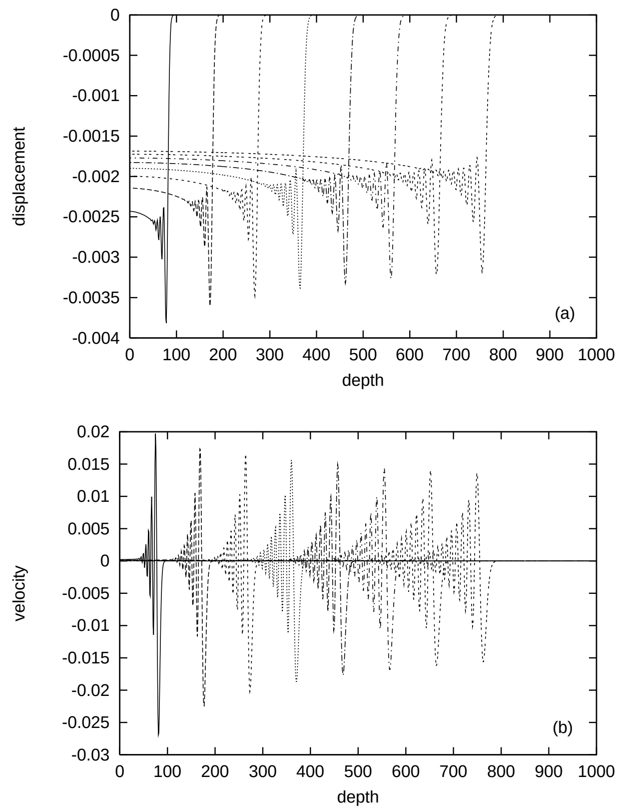

FIG. 1. (a) Snapshots of displacement of the propagating wave in a gravitationally compacted granular chain with Hertzian contact force law. Initial impulse is $v_i = 0.1$. (b) Snapshots of grain velocity corresponding to (a).

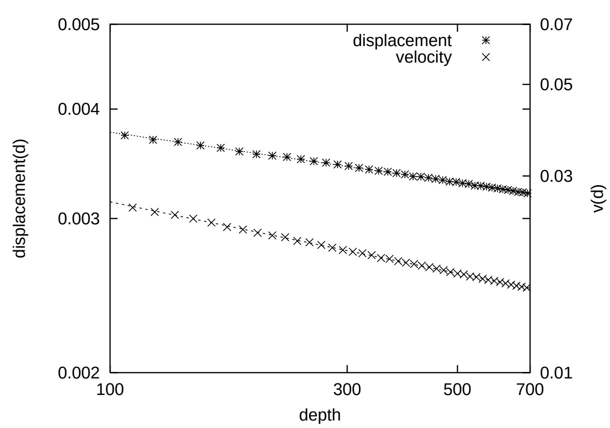

FIG. 2. Log_{10}-$\log_{10}$ plots of leading peaks of displacement and grain velocity shown in Fig. 1. Slopes of the straight lines are -0.0835 (displacement) and -0.250 (velocity), respectively.

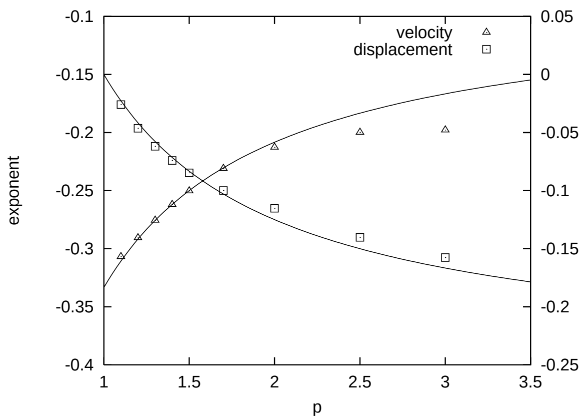

FIG. 3. Comparison of power-law exponents between theory and simulations for various values of p. Solid lines are theoretical results. Squares and triangles denote simulation data for displacement and grain velocity, respectively. Data are obtained for the leading peaks of each signal.

particles participating at the leading part of velocity signal, $N(h)$, which describes the wavelength. The power-law exponents of these quantities with error bounds are $T_{\max} \propto h^{0.170\pm0.002}, N \propto h^{0.338\pm0.004}$. These values are in good agreement with theoretical predictions for the Hertzian $p = 3/2$.

We obtain peak values of displacement and grain velocity signal for other values of p and plot them in Fig. 3. One can see a very nice fit to the theoretical curves up to $p = 2$. For large values of p, the deviation from theory occurs especially in grain velocity. This is understandable because nonlinearity becomes stronger as p increases and grain velocity contains more nonlinearity than displacement.

In conclusion, the propagating feature in vertical chain is dispersive due to gravity even though total energy and momentum are conserved. The effect of gravity induces the change in force constant at every contact. Therefore, the signal is no longer a soliton which is the propagating mode in horizontal chain [6,8]. We treat the problem analytically for the arbitrary power-law type of nonlinear contact forces and obtain general features of dispersive phenomena for a weak impulse in a gravitationally compacted chain. The normal mode solution for displacement has been obtained in the small oscillation and continuum limit. This normal mode solution describes the depth-dependent power law of displacement signal. We find various power laws describing signal characteristics depending on depth. This dispersive property is obtained by constructing both displacement and grain velocity function appropriately. We also perform numerical simulations for various power-law type contact force for a rather weak impulse $v_i = 0.1$ and show that the results agree well with our theoretical predictions.

The general features of soliton damping due to gravity may be given by studying similar work for a wide range of impulse, which will be given in a separate paper [12]. The properties of signal propagation studied in this work are fundamentals of the dynamics of granular chain under gravity. This work may be extended to higher dimensions and to more practical models for applications.

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