Projective Synchronization In Three-Dimensional Chaotic Systems

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In partially linear systems, such as the Lorenz model, chaotic synchronization is possible in only some of the variables. We show that, for the nonsynchronizing variable, synchronization up to a scale factor is possible. We explain the mechanism for this projective form of chaotic synchronization in three-dimensional systems. Projective synchronization is illustrated for the Lorenz and disk dynamo systems. We also introduce a vector field that can be used to predict the scaling factor. [S0031-9007(99)08897-3]

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If two identical copies of a chaotic system are started with similar initial conditions, their motions will not remain similar for long, for exponential divergence of orbits will amplify any initial small errors. It appears, at first, that it would be very difficult to keep both copies of a chaotic system synchronized. But, in 1990, Pecora and Carroll [1] showed that synchronization was indeed possible and, moreover, it could be achieved with a simple coupling. Since their work, the synchronization of chaotic dynamical systems has been intensively studied (see Ref. [2] for a recent review).

The basic idea in identical synchronization is to take two copies of a fixed chaotic system and let one control the other. The master (or drive) system provides a signal that is fed to the slave (or response) system. The signal is usually one of the coordinates of the master chaotic system. Synchronization can be thought of as a form of control of chaos and the simplicity of the coupling mechanism prompts many applications. Synchronization has been used as a method for transmitting a signal in a chaotic carrier [3–5], implementing with analog circuits a spread-spectrum transmitter. It has also been suggested as a method for repeating results in experimental chaos [6].

Consider the often studied example of synchronization, the Lorenz system [7],

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= (R - z)x - y, \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

The values of the parameters are set to $\sigma = 10.0$, $b = 8/3$, and $r = 60.0$. The Lorenz system exhibits absolute synchronization in the x coordinate, synchronization in the y coordinate, while z is not a synchronizing coordinate at all [8].

We will show, however, that even driving the response system with the z coordinate leads to synchronization, provided the definition is slightly modified. For a synchronizing system, one expects that the drive and response vectors tend to the same value. In *projective synchronization* the drive and response vectors synchronize up to a scaling factor—the vectors become proportional.

We have observed projective synchronization in partially linear systems, such as the Lorenz system. Partially linear systems are defined by a set of ordinary differential equations where the state vector can be broken into two parts $(\mathbf{u}, z)$. The equation for z is nonlinearly related to the other variable, while the equation for the rate of change of the vector $\mathbf{u}$ is linearly related to $\mathbf{u}$ through a matrix M that can depend on the variable z, as in

$$\begin{aligned} \dot{\mathbf{u}} &= M(z) \cdot \mathbf{u}, \\ \dot{z} &= f(\mathbf{u}, z). \end{aligned} \tag{2}$$

As with the Lorenz system, let M be a 2×2 matrix with components $m_{ij}(z)$ that are smooth functions and have no $\mathbf{u}$ dependence. For identical synchronization we will consider two copies of the system (2). One of the copies is the master system and evolves independently of the slave system. The two systems are coupled through z: The z in the slave system will be the z of the master system. The resulting system is a set of five differential equations:

$$\begin{aligned} \dot{\mathbf{u}}_m &= M(z) \cdot \mathbf{u}_m, \\ \dot{z} &= f(\mathbf{u}_m, z), \\ \dot{\mathbf{u}}_s &= M(z) \cdot \mathbf{u}_s, \end{aligned} \tag{3}$$

where $\mathbf{u}_m = (x_m, y_m)$ is the two-dimensional drive vector, and $\mathbf{u}_s = (x_s, y_s)$ is the response vector. With this notation, two systems are in projective synchronization when for an initial condition $\mathbf{u}_s(0)$ there is a constant α such that asymptotically in time

$$\|\alpha \mathbf{u}_m - \mathbf{u}_s\| \to 0. \tag{4}$$

The constant α could be negative and, as we will see later, depends in a simple way on the initial condition $\mathbf{u}_s(0)$.

Phase synchronization [9,10], often observed in the Rössler system [11], appears similar to projective synchronization. But there is a difference. In phase synchronization, the amplitudes remain chaotic and, in general, are

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uncorrelated [12], while, in projective synchronization, the amplitudes tend to some fixed ratio [see (16) below]. We do not study the Rössler system, as it is not partially linear.

Two physically important examples of partially linear systems are the Lorenz system (1), and the disk dynamo,

$$\begin{aligned} \dot{x} &= zy - \mu x\,, \\ \dot{y} &= (z - \gamma)x - \mu y\,, \\ \dot{z} &= 1 - xy\,. \end{aligned} \tag{5}$$

The Lorenz system was originally derived as a three-mode truncation of the equations describing the convection in a fluid layer [13], and it was later found to be similar to the model that describes the pulsations of a single-mode laser [14]. The dual-disk dynamo model [15] is a variation of the disk dynamo model [16] proposed to explain the essence of the mechanism governing the reversals of the earth's magnetic field. The Lorenz (1) and the dynamo (5) systems can each be coupled through the z variable, as in Eq. (3). However, this does not lead to identical synchronization as neither the x nor y coordinates of the drive or response system tend to each other. Examining the plot in Fig. 1 suggests that the ratio of corresponding coordinates approaches a constant, even though the initial conditions for the drive and response systems were different and not collinear.

The dynamo system has two fixed points that can be computed analytically. The linearization of the flow around these two fixed points has one negative eigenvalue and two eigenvalues with zero real part. We consider only the parameter values $\mu = 1.7$, $\gamma = 0.5$ for which the pair of fixed points is at $(x_0, y_0, z_0) = (1.968, \pm 0.929, \pm 1.076)$. The projection of the attractor onto the x-y plane is depicted in Fig. 2. Like the Lorenz model, two identical copies will be in projective synchronization when coupled through the z variable.

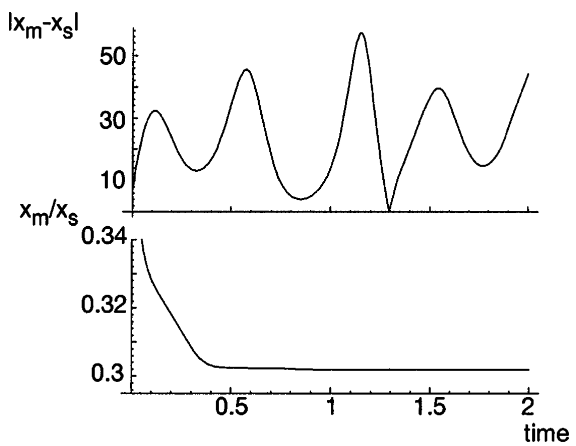

FIG. 1. Behavior of the drive (x_m) and response (x_s) subsystems in the Lorenz model. The difference does not settle down, while the ratio does. Similar behavior is seen for the y coordinate.

We will now explain the mechanism for projective synchronization of partially linear systems. The three-dimensional phase space of our system can be viewed as foliated by parallel planes $P_z = \{z = \text{constant}\}$ with a linear vector field M_z on each of them. (Let us fix the convention that subscript z means z is held fixed.) It is important that this vector field is common to both the drive system ($\mathbf{u}_m$) and the response system ($\mathbf{u}_s$).

The evolution of the system can be described in cylindrical coordinates (r, θ, z), with

$$(\dot{r}, \dot{\theta}, \dot{z}) = \left(\frac{1}{r}(x\dot{x} + y\dot{y}), \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}, \dot{z}\right). \tag{6}$$

From the equation of motion (2), the angle θ in the x-y plane evolves according to

$$\begin{aligned} \dot{\theta} = g_z(\theta) &\stackrel{\text{def}}{=} m_{21}\cos^2\theta - m_{12}\sin^2\theta \\ &\quad - (m_{11} - m_{22})\cos\theta\sin\theta\,. \end{aligned} \tag{7}$$

There are two equations similar to this one: one for the drive system and one for the response system. By subtracting them we get the evolution of the difference between the polar angles of the drive and response systems,

$$-\frac{\dot{\theta}_m - \dot{\theta}_s}{\sin(\theta_m - \theta_s)} = \frac{m_{12} + m_{21}}{\sec(\theta_m + \theta_s)} + \frac{m_{11} - m_{22}}{\csc(\theta_m + \theta_s)}\,. \tag{8}$$

Notice there is no radial dependence and the difference between the two angles occurs in factorized form. Those are consequences of the partial linearity of the system (2).

Because of ergodicity, the angle difference will at one point become small, at which time we can approximate the expression (8) by using the small angle variation $\omega = \theta_m - \theta_s$, which satisfies

$$\dot{\omega} = g_z(\theta_m) - g_z(\theta_s) = g_z'(\theta_m)\omega + \mathrm{O}(\omega)\,. \tag{9}$$

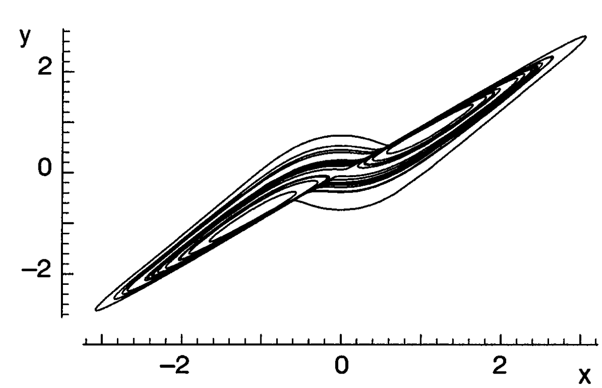

FIG. 2. Projection of the attractor of the disk dynamo onto the x-y plane.

For the Lorenz system, Eq. (9) reads

$$\dot{\omega} = -\omega[(R - z + \sigma)\sin(2\theta) + (\sigma - 1)\cos(2\theta)], \tag{10}$$

while for the dynamo system it is even simpler,

$$\dot{\omega} = -\omega(2z - \gamma)\sin(2\theta). \tag{11}$$

An ordinary differential equation, such as Eq. (9), has the general solution,

$$\omega(t) = \omega(0)\exp\left(\int_0^t g_z'[\theta(\xi)]\,d\xi\right). \tag{12}$$

If, for any constant $C > 0$, the integral

$$\int_0^t g_z'[\theta(\xi)]\,d\xi < -C \tag{13}$$

for sufficiently large values of the time t, then the difference ω between the drive and response systems will go to zero and the system will projectively synchronize.

The sign of the integral (13) is given by the sign of $g_z'[\theta(t)]$. In turn, the sign of g_z' is determined by the trigonometric functions in (10) or (11). For the Lorenz system, the first and third quadrants of the x-y plane contain most of the attractor. In those quadrants, $\sin 2\theta$ is positive and the first term in (10) dominates the second term, making g_z' negative most of the time. For the dynamo system (5), we see from Fig. 2 that the right-hand side of (11) will seldom change sign.

In Fig. 3, we plot $\log|\omega|$ and the values of $g_z'[\theta(t)]$. The value of $\log|\omega|$ decreases when g_z' is negative and increases when it is positive. One can also see that g_z' is negative more often than it is positive. This makes it clear that it is the left side of (9) that causes projective synchronization, that is, the asymptotic vanishing of the angle difference ω.

Let us also note that the functions $m_{ij}(z)$ need not be affine as is the case for both (1) and (5). For example, we can still observe projective synchronization for a modification of the Lorenz system, where the second equation reads

$$\dot{y} = (R - z^2)x - y. \tag{14}$$

Knowing how the angles synchronize, we investigate what happens to the radii. The time variation of the ratio of the radii

$$\frac{d}{dt}\frac{r_s}{r_m} = \frac{r_s}{r_m}\left(\frac{\dot{r}_s}{r_s} - \frac{\dot{r}_m}{r_m}\right) \tag{15}$$

is a function of $\dot{r}/r$. From partial linearity, $\dot{r}/r$ is a function only of the polar angle, and the quantity in the parentheses tends to zero as the polar angles become identical. Therefore, the limit

$$|\alpha| = |\alpha(x_s, y_s)| = \lim_{t\to\infty}\frac{r_s}{r_m} \tag{16}$$

exists for any initial condition (x_m, y_m, z_m) of the drive system. That means that the evolution of the response system $\mathbf{u}_s$ is asymptotically a scalar multiple of the evolution of the drive system, as seen in Fig. 4. We remark that the constant α can be negative, since the polar angle θ in (6) is determined up to a multiple of π.

Partial linearity fixes the value of α. Fix a plane P_z and in it an initial condition $\mathbf{u}_m = (x_m, y_m)$ of the drive system. The system (3) has the following property: Let $\mathbf{U}_1 = (\mathbf{u}_m, z, \mathbf{u}_s^1)$ and $\mathbf{U}_2 = (\mathbf{u}_m, z, \mathbf{u}_s^2)$ be two solutions. Then for any scalars a and b the vector function $\mathbf{U} = a\mathbf{U}_1 + b\mathbf{U}_2$ is also a solution of (3).

This implies the limit α is a linear function of $\mathbf{u}_s = (x_s, y_s)$ and has the form $\alpha = \mathbf{b}\cdot\mathbf{u}_s(0)$. The vector $\mathbf{b} \in R^2$ depends on the initial condition (x_m, y_m, z_m) of the

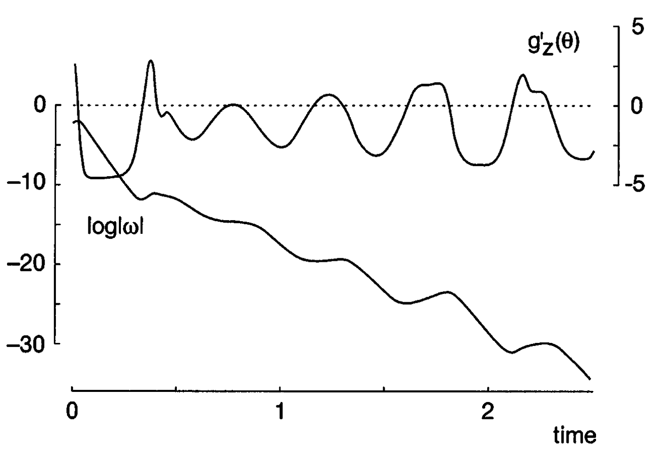

FIG. 3. Whenever the function g_z' is negative, the angle difference ω between the drive and response subsystems decreases as a function of time t. Plotted are the results for the Lorenz model.

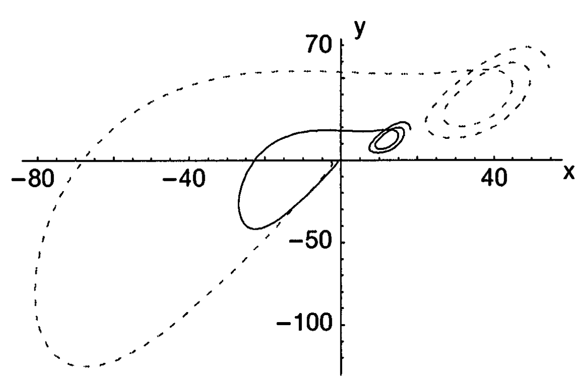

FIG. 4. Projection onto the x-y plane of the drive and response systems (Lorenz model). The initial conditions are close to the origin.

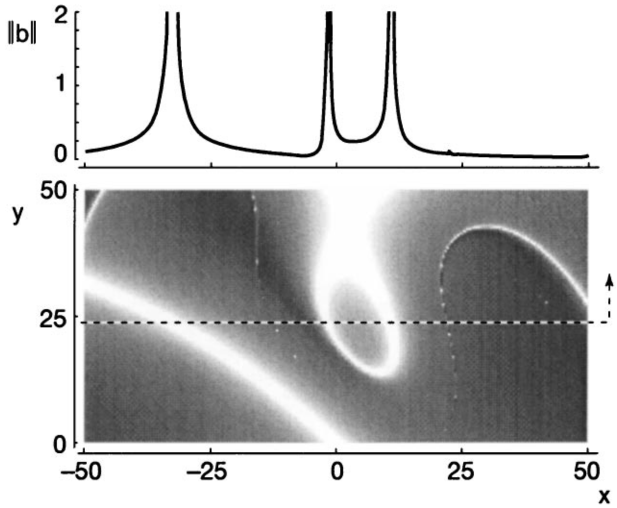

FIG. 5. The field $\mathbf{b}_z$ used to compute the scaling factor α for many initial conditions of the Lorenz model. Plotted is the $\|\mathbf{b}_{40}\|$, with lighter shades representing larger values. The field has the same symmetries as the Lorenz model. A cross section through $y = 24$ is also shown.

drive system and so is a map of $R^3 \rightarrow R^2$. It is useful to imagine $\mathbf{b}$ as a two-dimensional vector field defined on each plane P_z. To stress this viewpoint, we adopt the notation $\mathbf{b} = \mathbf{b}_z(\mathbf{u}_m)$. Thus, for each initial condition of the drive system $\mathbf{u}_m$ on a given plane P_z, the evolution of the response system (in particular, the magnitude and sign of α) is fully determined by the value of $\mathbf{b}_z(\mathbf{u}_m)$.

It is difficult to get any analytic information about the vector field $\mathbf{b}_z(\mathbf{u}_m)$ other than the trivial observation $\mathbf{b}_z(\mathbf{u}_m) \cdot \mathbf{u}_m = 1$, which follows from the fact that if we pick the same initial condition for both the drive and response systems, then indeed we get $\alpha = 1$.

However, there is an interesting consequence of $\alpha = \mathbf{b} \cdot \mathbf{u}_s$ in the plane P_z: For each initial condition $\mathbf{u}_m = \mathbf{u}_m(0)$ of the drive systems there exist a line of initial conditions $\mathbf{u}_s = (x_s, y_s) = \lambda \mathbf{b}_z^{\perp}(\mathbf{u}_m)$, $\lambda \in \mathbf{R}$ for which $\alpha = 0$. Thus $\|\mathbf{u}_s\| = 0$ along a one-dimensional subspace $N \subset P_z$. Consequently, there is a line through the initial condition $\mathbf{u}_m$ (namely, $\mathbf{u}_m + N$) such that, if we pick an initial condition of the response system on this line, the system synchronizes in the usual sense, that is, $\alpha = 1$.

There are regions in the plane P_z where $\alpha \gg 1$. This happens when the drive system dwells near the z axis, while the trajectory of the response system is pushed away by the unstable part of the linearized dynamics M_z. In the original system, this is not possible, because the z dynamics depends directly on x and y. In the response system, however, this self-control mechanism fails and the response variables may become arbitrarily large. In particular, for $\mathbf{u}_m$ lying on the stable manifold of the origin, the vector field $\mathbf{b}_z(\mathbf{u}_m)$ diverges. This can be seen in Fig. 5, where the norm of $\mathbf{b}_{40}$ is plotted.

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