Statistical Mechanics of an Oscillator Associative Memory with Scattered Natural Frequencies

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(Received 5 August 1998)

Analytic treatment of a nonequilibrium random system with large degrees of freedom is one of the most important problems of physics. However, little research has been done on this problem as far as we know. In this paper, we propose a new mean field theory that can treat a general class of nonequilibrium random system. We apply the present theory to an analysis of an associative memory with oscillatory elements, which is a well-known typical random system with large degrees of freedom. [S0031-9007(99)08816-X]

PACS numbers: 87.10.+e, 05.90.+m, 89.70.+c

The analytic treatment of a nonequilibrium random system with large degrees of freedom is one of the most important problems of physics. However, little research has been done on this problem as far as we know. In this paper, we propose a new mean field theory that can treat a general class of a nonequilibrium random system. We apply the present theory to an analysis of an associative memory with oscillatory elements, which is a well-known typical random system with large degrees of freedom.

Historically, there are two important studies regarding the nonequilibrium systems with large degrees of freedom that cannot be treated by conventional equilibrium statistical mechanics, since we cannot define the Lyapunov function with “bottoms.” Kuramoto [1] theoretically analyzed the mutual entrainment of uniformly coupled oscillators with scattered natural frequencies (Kuramoto theory). His model corresponds to a mean field model of a ferromagnet in equilibrium statistical mechanics. Kuramoto utilized the ideas of statistical mechanics, namely, a notion of macroscopic order parameters, to investigate his nonequilibrium system with large degrees of freedom. Daido [2] numerically analyzed the quasientrainment of randomly coupled oscillators with scattered natural frequencies. His model corresponds to the Sherrington-Kirkpatrick (SK) model of a spin-glass [3] in equilibrium statistical mechanics. A mean field theory should be developed for nonequilibrium random systems *with* frustration that follows the history of mean field theories in the equilibrium statistical mechanics. However, this kind of nonequilibrium random system has not yet been theoretically analyzed (see [4] for the random system *without* frustration). This is a famous “open problem” in physics [5]. That is the reason why we have proposed the present theory.

On the other hand, the Lyapunov function cannot be defined in spinlike systems with the nonmonotonic reaction function either. In the self-consistent signal-to-noise analysis (SCSNA) [6], the notion of a macroscopic order parameter is also introduced to analyze these frustrated spinlike systems, which cannot be treated by conventional equilibrium statistical mechanics. Note that the results of applying the SCSNA to simple random spin systems [6,7], i.e., the SK model and the Hopfield model [8], coincide with those of the replica theories [9,10].

In this paper, we propose a new theoretical framework for an oscillator associative memory model with scattered natural frequencies in memory retrieval states. This system can be considered as a typical example of nonequilibrium random systems with large degrees of freedom. The present theory makes a bridge between the SCSNA and the Kuramoto theory. Using the same procedure, we can easily treat a glass oscillator system [2]. Our theory is reduced to the Kuramoto theory in the finite loading case. When all oscillators have uniform natural frequencies, our theory coincides with the previously proposed theories [11,12] in the equilibrium statistical mechanics for an *XY* spin system.

The mutual entrainment is an important notion of a nonequilibrium system with large degrees of freedom. In uniformly coupled oscillators, there is a unique stable state, the ferromagnetic phase in the phase space. On the other hand, in frustrated systems, there are many stable states in the phase space. We need to elucidate the properties of the mutual entrainment in each stable phase (ferromagnetic phase and glass phase). Our new theory describes a phenomenon of the mutual entrainment in the ferromagnetic phase (memory retrieval). Thus, we numerically study a degree of the mutual entrainment in the glass phase (spurious memory retrieval). It is numerically shown in this paper that almost all oscillators synchronize under memory retrieval, but desynchronize under spurious memory retrieval when setting optimal parameters. Thus, it is possible to determine whether the recalling process is successful or not using information about the synchrony/asynchrony.

In general, when the coupling is sufficiently weak, the high-dimensional dynamics of a coupled oscillator system

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can be reduced to the phase equation [1,13]. Let us consider the following simplified model:

$$\frac{d\phi_i}{dt} = \omega_i + \sum_{j=1}^{N} J_{ij} \sin(\phi_j - \phi_i + \beta_{ij}), \qquad (1)$$

where N is the total number of oscillators, ϕ_i is the phase of the ith oscillator, and ω_i is the natural frequency assumed to be randomly distributed over the whole population with a density denoted by the symmetric distribution $g(\omega)$, i.e., $g(\omega) = g(-\omega)$. Note that the average of ω_i may be set to zero without a loss of generality. The theory presented below can be easily extended to treat the system with any other distribution $g(\omega)$. J_{ij} and β_{ij} denote an amplitude of a synaptic weight and a synaptic delay, respectively. In order to investigate the nature of frustrated nonequilibrium systems, we have selected the following generalized Hebb learning rule [11] to determine J_{ij} and β_{ij}:

$$C_{ij} = J_{ij} \exp(i\beta_{ij}) = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \overline{\xi}_j^\mu, \qquad \xi_i^\mu = \exp(i\theta_i^\mu), \qquad (2)$$

as their typical example. $\{\theta_i^\mu\}_{i=1,\dots,N,\mu=1,\dots,p}$ are the phase patterns to be stored in the network and are assigned to random numbers with a uniform probability in $[0, 2\pi]$. Here, we define a parameter α (loading rate) such that $\alpha = p/N$. In the equilibrium limit of this model, that is, $g(\omega) = \delta(\omega)$, the storage capacity given by $\alpha_c = 0.038$ [12].

We put $s_i = \exp(i\phi_i)$ for the sake of simplicity. The order parameter m^μ, which measures the overlap between the system state s_i and the embedded pattern $\boldsymbol{\xi}^\mu$, is defined as

$$m^\mu = \frac{1}{N} \sum_{j=1}^{N} \overline{\xi}_j^\mu s_j. \qquad (3)$$

We obtain order parameter equations of the present system by applying the following manipulations: First, assuming a self-consistent local field for each of the oscillators, a distribution of s_i under $g(\omega_i)$ is formally derived by the Kuramoto theory. Second, we estimate the contribution of randomness, that is, the uncondensed patterns in the present case by the SCSNA and determine the local field in a self-consistent manner. Finally, the order parameter equations are obtained using the self-consistent local field. A detailed derivation of the present theory will be discussed elsewhere. Here, we assume $m^1 = O(1)$ and $m^\mu = O(1/\sqrt{N})$ for $\mu > 1$ (uncondensed patterns). Then, the following two dimensional equations for the order parameters are obtained:

$$m = \langle\langle X(x_1, x_2, \xi) \rangle\rangle_{x_1, x_2, \xi}, \qquad (4)$$

$$U = \langle\langle F_1(x_1, x_2, \xi) \rangle\rangle_{x_1, x_2, \xi}, \qquad (5)$$

where $\langle\langle \cdots \rangle\rangle_{x_1,x_2,\xi}$ is taken to mean the Gaussian average over x_1, x_2 and condensed pattern ξ^1, $\langle\langle \cdots \rangle\rangle_{x_1,x_2,\xi} = \langle \int \int Dx_1\, Dx_2 \cdots \rangle_\xi$. The pattern superscripts 1 of m are omitted for brevity. The self-consistent mean field $\tilde{h}$, X, F_1 and the Gaussian measure $Dx_1 Dx_2$ are expressed as follows:

$$Dx_1 Dx_2 = \frac{dx_1 dx_2}{2\pi\rho^2} \exp\left(-\frac{x_1^2 + x_2^2}{2\rho^2}\right), \qquad (6)$$

$$\rho^2 = \frac{\alpha}{2(1-U)^2}, \qquad \tilde{h} = \xi m + x_1 + i x_2, \qquad (7)$$

$$X(x_1, x_2, \xi) = \tilde{h} \int_{-1}^{1} dx\, g(|\tilde{h}|x) \sqrt{1 - x^2}, \qquad (8)$$

$$F_1(x_1, x_2, \xi) = \int_{-1}^{1} dx \left(g(|\tilde{h}|x) + \frac{|\tilde{h}|}{2} x g'(|\tilde{h}|x) \right) \times \sqrt{1 - x^2}. \qquad (9)$$

Here, U corresponds to the susceptibility, which measures the sensitivity to external fields. A distribution of resultant frequencies $\overline{\omega}$ in the memory retrieval state, which is denoted as $p(\overline{\omega})$, becomes

$$p(\overline{\omega}) = r\delta(\overline{\omega}) + \left\langle \int Dx_1\, Dx_2 \frac{g(\overline{\omega}\sqrt{1 + \frac{|h|^2}{\overline{\omega}^2}})}{\sqrt{1 + \frac{|h|^2}{\overline{\omega}^2}}} \right\rangle_\xi, \qquad (10)$$

$$r = \left\langle \int Dx_1\, Dx_2 |\tilde{h}| \int_{-1}^{1} dx\, g(|\tilde{h}|x) \right\rangle_\xi, \qquad (11)$$

where r measures the ratio between the number of synchronous oscillators and the system size N. We now consider the relationships between the present theory and the previously proposed theories. For the equilibrium limit, $g(x) = \delta(x)$, we obtain

$$X = \frac{\tilde{h}}{|\tilde{h}|}, \qquad F_1 = \frac{1}{2|\tilde{h}|}, \qquad p(\overline{\omega}) = \delta(\overline{\omega}), \qquad (12)$$

which coincide with the replica theory [12] and the SCSNA [11]. On the other hand, regarding the uniform-system limit, $\alpha = 0$, our theory reproduces the Kuramoto theory as

$$m = m \int_{-1}^{1} dx\, g(|m|x) \sqrt{1 - x^2}, \qquad (13)$$

$$p(\overline{\omega}) = r\delta(\overline{\omega}) + \frac{g(\overline{\omega}\sqrt{1 + \frac{|m|^2}{\overline{\omega}^2}})}{\sqrt{1 + \frac{|m|^2}{\overline{\omega}^2}}}, \qquad (14)$$

$$r = |m| \int_{-1}^{1} dx\, g(|m|x). \qquad (15)$$

As mentioned before, our theory has made a bridge between the equilibrium-frustrated system and the non-equilibrium-uniform system. In addition, we have presented a systematic way of analytical treatments for the nonequilibrium random systems. If C_{ij} is assigned to random numbers with a Gaussian in a complex plane, $\mathrm{Re}[C_{ij}] \sim \mathcal{N}(1/N, \alpha/2N)$, $\mathrm{Im}[C_{ij}] \sim \mathcal{N}(0, \alpha/2N)$, the order parameter equation consists with Eq. (4) under

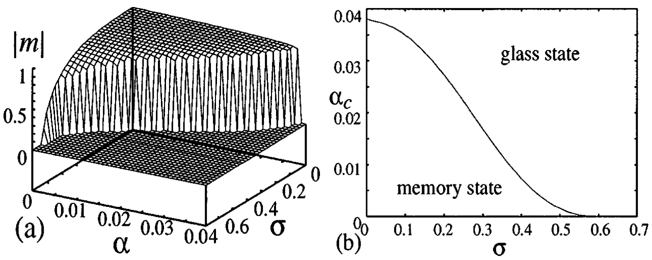

FIG. 1. (a) Phase diagram in $(|m|, \sigma, \alpha)$ space. (b) σ vs critical memory capacity α_c.

the constraint $U = 0$. The detailed results will be presented elsewhere. We can also treat Daido's glass oscillator system (with real number interaction) [2] using the same procedure.

In the following analyses, we choose a system with $g(\omega) = (2\pi\sigma^2)^{-1/2} \exp(-\omega^2/2\sigma^2)$. Figure 1(a) shows a phase diagram in the $(|m|, \sigma, \alpha)$ space, which was obtained by numerically solving these order parameter equations. A cross section of this curved surface at $\sigma = 0$ coincides with the results of the SCSNA [11] and the replica theory [12]. Furthermore, a cross section of this curved surface at $\alpha = 0$ is equal to a result of the Kuramoto theory [1]. Thus, our theory bridges the gap between these theories. Figure 1(b) shows the values of critical memory capacity α_c for various values of σ. α_c decreases monotonically as σ increases. The critical value of σ at $\alpha_c = 0$ is given by $\sigma_c = 0.62$, which coincides with that of the Kuramoto theory. Figures 2(a)–2(d) display $|m|$ values vs σ for various values of α, where the solid curves are obtained theoretically, and the plots show results obtained by numerical simulation. According to these figures, the theory is in good agreement with the simulation results.

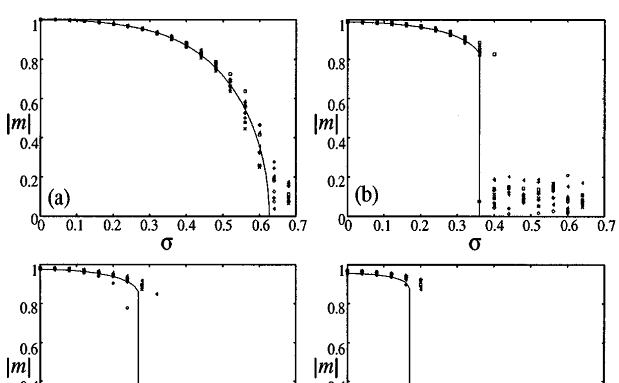

FIG. 2. Values of $|m|$ vs σ (solid curves theoretically obtained, and plots obtained by numerical simulation). (a) $\alpha = 0.0$ ($p = 1$, $N = 1000$). (b) $\alpha = 0.01$ ($N = 2000$). (c) $\alpha = 0.02$ ($N = 2000$). (d) $\alpha = 0.03$ ($N = 2000$).

Next, we examined the distributions of the resultant frequencies $\overline{\omega}_i$ over the whole population of oscillators in the memory retrieval state and the spurious memory state. Here, the resultant frequencies $\overline{\omega}_i$ were calculated by using the long time average of $d\phi_i/dt$. The plots in Figs. 3(a) and 3(b) show the values of resultant frequencies $\{\overline{\omega}_i\}_{i=1,\ldots,N}$ vs σ, that is, a bifurcation diagram. Figure 3(a) denotes $\overline{\omega}_i$ distributions in memory retrieval states, and Fig. 3(b) represents those in spurious memory states. These results show that there exists a region σ that satisfies the two conditions: all oscillators mutually synchronize in memory retrieval, and oscillators desynchronize in spurious memory retrieval. Thus, it is possible to determine whether the recalling process is successful or not using only the information about the synchrony/asynchrony, when proper σ is given. In Figs. 3(a) and 3(b), all phase values ϕ_i continued drifting toward negative directions due to an offset of scattered ω_i. In other samples, all ϕ_i continued drifting toward positive directions.

We investigated the effect of the system size N on the $\overline{\omega}_i$ distribution. The solid curves in Fig. 4 indicate the distributions of the resultant frequencies $p(\overline{\omega})$ in Eq. (10). The histograms in Fig. 4 show the results from the numerical simulation. As shown in Figs. 4(a)–4(a″), the theory agrees well with the simulation results in memory retrieval states. As shown in Figs. 4(b)–4(b″), there exists a delta peak that indicates the mutual entrainment of a large population of oscillators in memory retrieval states. However, in spurious memory states, the distribution of the average frequency is gentle compared to the distribution in memory retrieval states, which indicates the quasientrainment of oscillators [2]. According to these figures, we believe the phenomena mentioned before are invariant to the system size N. The reason why all oscillators mutually synchronize in memory retrieval states, but desynchronize in spurious states, is because effective interactions among oscillators in memory states (as in ferromagnetic states [1]) are different from interactions in spurious memory states (as in spin-glass states [2]), in which the system is strongly frustrated.

The phase description proposed here can be considered as a minimum model of neural networks based on

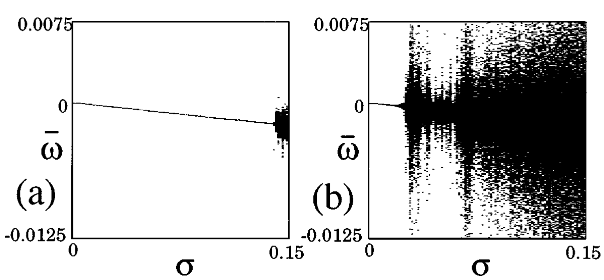

FIG. 3. Values of resultant frequencies $\{\overline{\omega}_i\}_{i=1,\ldots,N}$ vs σ, $N = 2000$, and $\alpha = 0.0315$. (a) Memory pattern retrieval. (b) Spurious memory pattern retrieval.

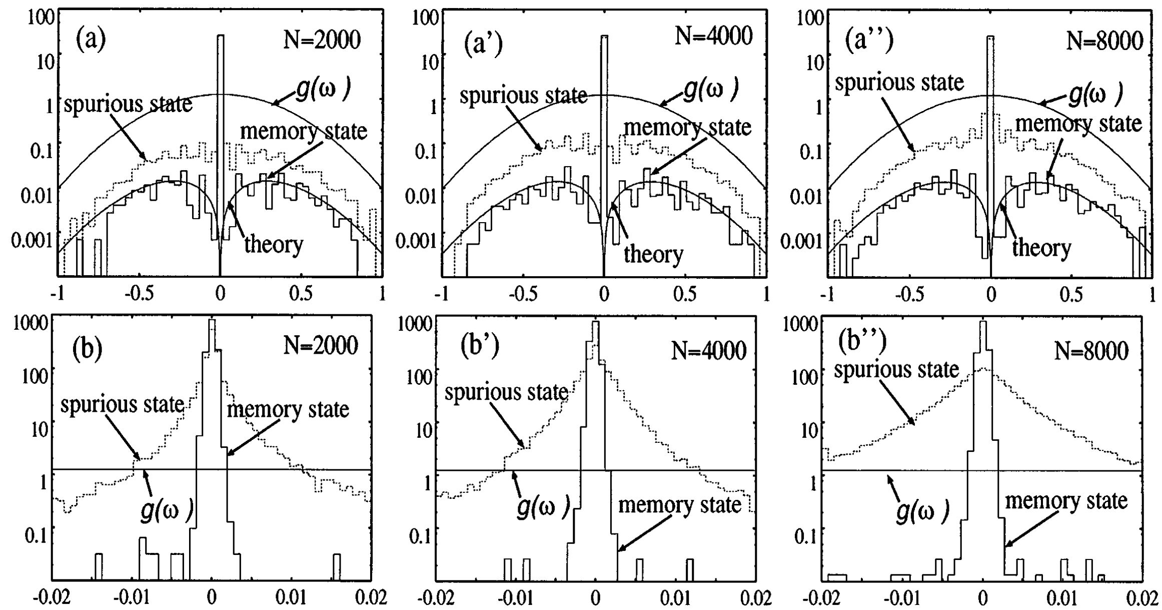

FIG. 4. Distribution of resultant frequencies state. $\alpha = 0.01$; $\sigma = 0.32$. Solid curves show theoretical results of $p(\overline{\omega})$ in Eq. (10), but delta peaks at $\overline{\omega} = 0$ are not indicated. (b),(b′), and (b″) display detailed distributions of (a),(a′), and (a″) at $\overline{\omega} = 0$, respectively. (a),(b) $N = 2000$. 20 trials. (a′),(b′) $N = 4000$. 12 trials. (a″),(b″) $N = 8000$. 12 trials.

oscillatory activities that is mathematically solvable. Chaos neural networks yield rich phenomena as discussed here, but cannot be easily analyzed, except with simulations. Since the present analysis corresponds to the replica symmetric approximation, we have noted that it should be extended to the replica symmetry breaking in order to properly treat the spurious states (spin-glass states), and this remains for a future paper.

In the field of neuroscience, a growing number of researchers have been interested in the synchrony of oscillatory neural activities because physiological evidence of their existence has been obtained in the visual cortex of a cat [14,15]. Much experimental and theoretical research exists regarding the functional role of synchronization. One of the more interesting hypotheses is called *synchronized population coding,* which was proposed by Phillips and Singer. However, its validity is highly controversial [16]. In this paper, we numerically showed the possibility of determining if the recalling process is successful or not using information about the synchrony/asynchrony. If we consider information processing in brain systems, the solvable *toy* model presented in this paper may be a good candidate for showing the validity of a synchronized population coding in the brain, and we believe the present analysis may strongly influence a debate on the functional role of synchrony.

The authors thank Ms. M. Yamana, Mr. M. Yoshioka, and Dr. M. Shiino for their valuable discussions. This work was partially supported by Grants-in-Aid for Scientific Research in Priority Area (2) No. 07252219, Grants-in-Aid for Encouragement of Young Scientists No. 0782, and the Japan Society for the Promotion of Science (JSPS).

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