Magnetic Dipole Equilibrium Solution at Finite Plasma Pressure

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A realistic equilibrium with finite plasma pressure is derived for a plasma confined by the magnetic field of a point dipole. The low and high pressure forms of the solution are explicitly displayed. The energy principle is used to demonstrate the interchange stability of the equilibrium solution for arbitrary pressures and shows that it remains stable as the plasma pressure increases. [S0031-9007(99)08794-3]

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Dipole confinement devices are axisymmetric toroidal systems in which the dipolar magnetic field is created by a current ring [1,2]. All other equilibrium currents are plasma currents in the toroidal direction—there are no parallel currents. All magnetic field lines are closed so that "flux" surfaces are defined as surfaces of rotation about the axis of the current ring by the closed field lines. These surfaces are also the surfaces on which the pressure is constant. Because of the dipole's geometrical simplicity, charged particles remain on flux surfaces in their guiding center motion: There are no banana orbits, and no neoclassical enhancements to classical transport [3]. Moreover, the diamagnetic toroidal flows have a vanishing divergence so no Pfirsch-Schülter flows are generated [3]. Of course, dipolar features are observed in planetary magnetospheres [4], so the interest in dipole confinement is not limited to the laboratory.

Remarkably, however, physically interesting and mathematically simple equilibrium solutions for a plasma confined by the magnetic field of a point dipole—that is, solutions of the relevant Grad-Shafranov equation [5] in the absence of inertial and gravitational effects [6]—are not available in the literature and have not been shown to exist. Recent work by Tur, Maurice, Blanc, and Yanovsky [7] attempts to remedy this situation by considering the limit of a point dipole and a pressure profile that is proportional to the flux squared so that the Grad-Shafranov equation reduces to a linear differential equation. Aside from the mathematical complexity of the resulting solution, the quadratic pressure profile assumption yields a toroidal plasma current density that does not decrease with the radial distance from the dipole. Even when their solution is matched to an external vacuum region, the resulting equilibrium is not physically appealing, because the small pressure limit of the solution is obtained as an expansion in the distance from the point dipole to the fourth power. Interestingly, however, their weak pressure solution contradicts earlier claims that the leading finite $\beta = (\text{plasma pressure})/(\text{magnetic pressure})$ modification to the vacuum dipole magnetic field is a change in the magnitude of the field, but not its direction [8].

To obtain an equilibrium dipole solution with a finite total plasma current, that is, a current density that decreases faster than the third power of the distance from the dipole, we consider separable solutions of the Grad-Shafranov equation,

$$\nabla \cdot \left(\frac{1}{R^2} \nabla \psi \right) = -4\pi \frac{dp}{d\psi}, \tag{1}$$

where $p = p(\psi)$ is the plasma pressure, and ψ is the flux function associated with the dipole magnetic field $\vec{B} = \nabla\psi \times \nabla\zeta$, with the ζ toroidal angle with respect to the dipole axis and R the cylindrical radial distance from the axis of the dipole. Equation (1) is obtained as usual from the steady state toroidal component of Ampère's law, $c\nabla \times \vec{B} = 4\pi\vec{J}$, and force balance, $\vec{J} \times \vec{B} = c\nabla p$. The flux function ψ is related to the toroidal component of the vector potential, A, by $\psi = RA$. We employ spherical coordinates r, θ, and ζ with $\mu = \cos\theta$, $R = r\sin\theta$, and seek separable solutions of the form,

$$\psi = \frac{H(\mu)}{r^\alpha}, \tag{2}$$

where H is an unknown function of μ alone that becomes a constant times $(1 - \mu^2)$ in the vacuum limit. The parameter α plays the role of an eigenvalue of the nonlinear Grad-Shafranov equation and is equal to unity in the vacuum limit to recover the vacuum dipole solution $\psi \propto \sin^2\theta/r$. Plasma currents will cause the parameter α to depart from unity, and, since the plasma current must be in the same direction as the dipole current for equilibrium, the finite plasma pressure acts to reduce α from unity as the plasma pressure increases, as will be shown shortly.

Inserting Eq. (2) into Eq. (1) and using spherical variables gives

$$\frac{\partial^2 H}{\partial \mu^2} + \frac{\alpha(\alpha + 1)}{(1 - \mu^2)} H = -4\pi \frac{dp}{d\psi} \left(\frac{H}{\psi} \right)^{1+4/\alpha} . \quad (3)$$

Therefore, for $H = H(\mu)$ only, we must assume $p \propto \psi^{2+4/\alpha}$. Employing a pressure profile of the form

$$p = p_0(\psi/\psi_0)^{2+4/\alpha}, \quad (4)$$

with p_0 the pressure at some reference surface ψ_0, and inserting

$$H(\mu) = \psi_0 R_0^{\alpha} h(\mu), \quad (5)$$

with R_0 the cylindrical radius at which the surface ψ_0 intersects the symmetry plane ($\theta = \pi/2$), yields the nonlinear Grad-Shafranov equation for the unknown function $h(\mu)$, which we write in a form particularly convenient at low plasma pressure,

$$\frac{d}{d\mu} \left[(1 - \mu^2)^2 \frac{d}{d\mu} \left(\frac{h}{1 - \mu^2} \right) \right] - (1 - \alpha)(2 + \alpha)h = -\beta_0 \alpha (2 + \alpha)(1 - \mu^2) h^{1+4/\alpha}, \quad (6)$$

where $h \to 1 - \mu^2$ as $\beta_0 \to 0$ and

$$\beta_0 = \frac{8\pi p_0}{B_0^2} = \frac{8\pi p_0 R_0^4}{\alpha^2 \psi_0^2} . \quad (7)$$

To define the plasma beta at the equatorial or symmetry plane, β_0, we use $\vec{B} = \nabla \times (\psi \nabla \zeta)$ to find the magnitude of the magnetic field at R_0 (the intersection of the symmetry plane and the reference surface ψ_0) to be $B_0 = \alpha\psi_0/R_0^2$ assuming $h(\mu = 0) = 1$. Notice for the pressure profile and flux function considered here, the local plasma beta, $\beta = 8\pi p/B^2$, is independent of the spherical radial variable r, so β is simply β_0 times a function of angle θ.

Using the boundary conditions that $\vec{B}$ be finite and parallel to the axis of symmetry at $\theta = 0$ and $\theta = \pi/2$,

$$h(\mu \to 1) = 1 - \mu, \qquad dh/d\mu|_{\mu=0} = 0, \quad (8)$$

integration of Eq. (6) from $\mu = 0$ to $\mu = 1$ gives

$$(2 + \alpha)[(1 - \alpha)P_1 - \beta\alpha P_2] = 0, \quad (9)$$

with

$$P_1 = \int_0^1 d\mu\, h, \qquad P_2 = \int_0^1 d\mu\, (1 - \mu^2) h^{1+4/\alpha}. \quad (10)$$

Because $h > 0$, to satisfy Eq. (9) α should either be in the range $0 < \alpha \leq 1$ or $\alpha = -2$ [for $\alpha = -2$, $p =$ const, and the solution to Eq. (6) is simply the vacuum solution $h = 1 - \mu^2$, which corresponds to the solution near the symmetry axis on the interior of a finite current ring generating a vacuum dipole magnetic field [9]]. Notice that for $0 < \alpha \leq 1$ the pressure peaks at the innermost flux surface, that is, at the location of the point dipole. However, for a finite current ring the pressure peaks at some distance from the ring before falling off. Consequently, our model is expected to be appropriate beyond a couple ring diameters.

Equation (9) indicates that the departure of α from unity is due to the finite β_0 of the plasma (which in turn modifies the pressure profile). Inserting the $\beta_0 \to 0$ result $h = 1 - \mu^2$ into Eq. (9) and assuming $\alpha \to 1$ gives the departure of α from unity to be of the order of β_0,

$$1 - \alpha = \frac{512}{1001} \beta_0 . \quad (11)$$

As a result, an analytic solution to Eq. (6) can be found for small β_0 by using the replacement $h \to 1 - \mu^2$ in the two terms in which h appears undifferentiated. Using the boundary condition (8) at $\mu = 1$ to integrate the resulting equation from μ to 1 and letting $t = 1 - \mu^2$ gives [10]

$$\frac{d}{dt}\left(\frac{h}{t}\right) = \frac{3/4}{t^2(1 - t)^{1/2}} \left[(1 - \alpha) \int_0^t \frac{dx\, x}{(1 - x)^{1/2}} - \beta_0 \int_0^t \frac{dx\, x^6}{(1 - x)^{1/2}} \right]$$
$$= \frac{192\beta_0}{1001} \left[1 + \frac{5}{6} t + \frac{35}{48} t^2 + \frac{21}{32} t^3 + \frac{77}{128} t^4 \right].$$

Integrating again, using $h(\mu = 0) = 1$, results in the following low β_0 solution valid at all distances from a point dipole:

$$\frac{h}{1 - \mu^2} = 1 - \frac{192\beta_0}{1001} \Big\{ [1 - (1 - \mu^2)] + \frac{5}{12}[1 - (1 - \mu^2)^2] + \frac{35}{144}[1 - (1 - \mu^2)^3] + \frac{21}{128}[1 - (1 - \mu^2)^4] + \frac{77}{640}[1 - (1 - \mu^2)^5] \Big\}, \quad (12)$$

where, of course, $1 - \mu^2 = \sin^2 \theta$.

The plasma current density $\vec{\mathrm{J}}$ is found from

$$\vec{\mathrm{J}} = J\hat{\zeta} = \hat{\zeta} cR \frac{dp}{d\psi} = \hat{\zeta} \frac{c\beta_0(2+\alpha)B_0R_0^{\alpha+2}}{4\pi r^{\alpha+3}}(1-\mu^2)^{1/2}h^{1+4/\alpha}, \quad (13)$$

with $\hat{\zeta} = R\nabla\zeta$ the toroidal unit vector. Notice that at the midplane $J \propto 1/r^{3+\alpha}$ so the total current in the plasma, $\int dr\, d\theta rJ \propto \int dr/r^{2+\alpha}$, is always well behaved as $r \to \infty$ if $0 < \alpha \leq 1$. For $dp/d\psi \propto \psi$, the case considered by Tur, Maurice, Blanc, and Yanovsky [7], $J \propto R\psi \propto$ const, so the total plasma current increases as r^2.

The magnetic field associated with the flux function ψ,

$$\vec{\mathrm{B}} = \frac{B_0R_0^{\alpha+2}}{\alpha r^{\alpha+2}}\left\{\hat{\theta}\frac{\alpha h}{(1-\mu^2)^{1/2}} + \hat{r}\frac{dh}{d\mu}\right\}, \quad (14)$$

may also be evaluated, where $\hat{r} = \nabla r$ and $\hat{\theta} = r\nabla\theta$ are unit vectors. As in the solution of Tur, Maurice, Blanc, and Yanovsky, a finite plasma pressure changes the direction of the magnetic field as well as its magnitude, contradicting the work of Chan *et al.* [8].

Next, we consider the case of large β_0 ($\beta_0 \gg 1$) to demonstrate that solutions to the Grad-Shafranov equation exist for arbitrary beta equilibrium. The low pressure solution and the lower limit on the allowed α range suggest that α decreases toward zero as β_0 increases to infinity. Consequently, we assume that $1/\beta_0 \ll \alpha(\beta_0) \ll 1$. We will verify this assumption by showing that our solution requires $\alpha = \beta_0^{-1/2}$.

We begin by considering the Grad-Shafranov equation (6) in the form

$$\frac{d^2h}{d\mu^2} + \frac{\alpha(\alpha+1)}{1-\mu^2}h = -\beta_0\alpha(2+\alpha)h^{1+4/\alpha}, \quad (15)$$

where we need only consider $0 \leq \mu \leq 1$, since we are interested in a solution even in μ. When $\beta_0 \gg 1$, the term $\alpha(\alpha+1)h/(1-\mu^2)$ is small everywhere [recall Eq. (8)]. The term $\beta_0\alpha(\alpha+2)h^{1+4/\alpha}$ is large at $\mu = 0$ and rapidly decreases to zero as h decreases from $h(\mu = 0) = 1$ toward $h(\mu = 1) = 0$ since $\alpha \ll 1$. As a result, to lowest order we need only solve

$$\frac{d^2h}{d\mu^2} = -2\beta_0\alpha h^{1+4/\alpha}. \quad (16)$$

Multiplying by $dh/d\mu$ and integrating from $\mu = 0$, where $dh/d\mu = 0$ by symmetry, we find

$$\frac{dh}{d\mu} = -\alpha\beta_0^{1/2}(1-h^{2+4/\alpha})^{1/2} + O(\alpha). \quad (17)$$

Integrating from $\mu = 0$, where $h(\mu = 0) = 1$, to μ gives

$$\alpha\beta_0^{1/2}\mu = \int_h^1 dx\,(1-x^{2+4/\alpha})^{-1/2} \underset{\mu\to 1}{\longrightarrow} 1 - h, \quad (18)$$

where the $\mu \to 1$ form is valid for $\mu \gg \beta_0^{-1/2}$. To satisfy $h(\mu = 1) = 0$ requires

$$\alpha = 1/\beta_0^{1/2} + O(1/\beta_0). \quad (19)$$

Notice that for large β_0, $h = 1 - \mu$ everywhere except in a small region $0 \leq \mu \lessapprox \beta_0^{-1/2} \ll 1$, where h remains close to unity, but with a large second derivative of the order of $\beta_0^{1/2}$.

The preceding demonstrates that separable dipolar solutions to the Grad-Shafranov equation exist for arbitrarily large β_0. The distance between adjacent flux surfaces at the symmetry plane $\mu = 0$ increases as β_0 increases as can be seen by realizing that as α decreases the spacing must adjust to keep $\psi \propto (R_0/r)^\alpha$ fixed. As a result, the constant ψ surfaces become more extended and localized about the symmetry plane as β_0 increases. The resulting large β_0 equilibrium resembles the accretion disk associated with star formation [11].

Energy principle arguments are normally invoked to see that interchange (or flute) stability for an adiabatic plasma in a vacuum dipole magnetic field requires $20/3 > -(r/p)(dp/dr)$ [12]. For our pressure profile in the vacuum limit $p \propto \psi^6 \propto r^{-6}$, and we see that this stability condition is satisfied. More interestingly, we can demonstrate that finite beta effects enhance this interchange stability. We start with the necessary condition for finite beta interchange stability which can be written in the form

$$\left(\frac{1}{p}\frac{dp}{d\psi}\right)\left(\frac{1}{\nu}\frac{d\nu}{d\psi}\right) + \frac{5}{3}\left(\frac{1}{\nu}\frac{d\nu}{d\psi}\right)^2 > 0, \quad (20)$$

where ν is the volume per unit flux at fixed ψ, which for our solution becomes

$$\nu = \oint d\ell/B = \oint d\theta r/\hat{\theta}\cdot\vec{\mathrm{B}} = (R_0^3/\alpha\psi^{1+3/\alpha})\int_0^1 d\mu\, h^{1+3/\alpha}. \quad (21)$$

Using the preceding we see that $\nu \propto \psi^{-1-3/\alpha}$, while from Eq. (4) we have $p \propto \psi^{2+4/\alpha}$. As a result, Eq. (20) gives the finite beta modified interchange stability condition for our solution to be

$$\frac{5}{3} > \frac{2(2+\alpha)}{3+\alpha}. \quad (22)$$

Because α decreases from unity towards zero as β_0 increases from zero toward infinity, we see that the interchange stability is maintained at all plasma pressures.

To verify that Eq. (20) is valid for arbitrary beta, the energy principle for interchange modes in the absence of the parallel current is varied with respect to the parallel displacement to obtain the plasma compressibility term that arises because of the closed field lines [13]. Next, variations with respect to the two perpendicular displacements are performed to obtain the general finite beta interchange stability condition for arbitrary axisymmetric closed field line geometries:

$$\frac{2\gamma p\langle\vec{\kappa}\cdot\nabla\psi/R^2B^2\rangle}{1+4\pi\gamma p\langle B^{-2}\rangle} > \frac{dp}{d\psi}, \qquad (23)$$

where $\vec{\kappa}$ is the curvature, $\gamma = \frac{5}{3}$, and $\langle\cdots\rangle = \oint d\ell B^{-1}(\cdots)/\oint d\ell B^{-1}$. Notice that closed field lines result in plasma compressibility acting to make curvature a stabilizing influence for interchange modes. When Eq. (23) is rewritten using perpendicular pressure balance and the Grad-Shafranov equation, Eq. (20) is recovered.

For short wavelength ballooning modes the stabilizing influence of plasma compressibility is lost and replaced by the stabilizing influence of line bending. Whether there is a critical beta beyond which our equilibrium becomes unstable to ballooning modes is at present unclear, because as beta increases the destabilizing bad curvature regions become more localized about the intersection of the flux surface with the equatorial plane, where the rapid variation of the displacement makes line bending strongly stabilizing. Further details of the interchange and high mode number ballooning stability analysis for our finite beta equilibrium are beyond the scope of the present treatment and will be published elsewhere. Of course, more sophisticated stability analyses are required to determine low mode number ballooning [14] and drift [15] stability of our separable solution. Other stability issues that remain unclear are the effects of departures from our exact equilibrium solution due to error fields, for example, due to imperfect coil alignment in the laboratory or asymmetries introduced by the solar wind in space physics applications, or due to the presence of a toroidal magnetic field, as might be generated by the rotation expected in stellar accretion disks.

In summary, we have shown that a physically realistic, finite plasma pressure solution of the Grad-Shafranov equation exists for a point dipole magnetic field configuration, explicitly evaluated the low and high β_0 forms of the solution, demonstrated that general β_0 solution is interchange stable for arbitrary beta, and that the interchange stability is maintained as β_0 increases.

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