Partially Coherent Solitons on a Finite Background

Nail Akhmediev and Adrian Ankiewicz
Australian Photonics CRC, Optical Sciences Centre, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia
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We have found a new class of exact solutions for multicomponent partially coherent solitons in photorefractive media with a drift mechanism of nonlinear response. Their novel feature is that the composite M-component sech-shaped soliton is located on a constant-background plane wave. The solutions are characterized by two free parameters and these are related to the maximum intensity of the composite soliton and to the amplitude of the background. [S0031-9007(99)08778-5]

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The concept of incoherent solitons, both in time [1] and in space [2,3], has recently attracted considerable attention [4], especially since the first experimental demonstration of their existence [5]. Photorefractive media are the most appropriate for experimental observations, as they require only extremely low optical powers [6,7]. Theoretical descriptions of incoherent solitons have been presented based on various principles. Approaches based on multimode waveguides [8] and the geometric optics limit [9,10] are useful tools for understanding the idea of incoherent solitons. As a further conceptual step, it has been shown that these novel objects have variable shape and that they can be reshaped after collisions [11]. On a descriptive level, the difference between a standard single soliton and an incoherent soliton could be compared to that between an elementary particle and a complicated structure, such as an atom. Indeed, detailed analysis has shown that partially coherent solitons (PCS) are multiparameter families of solutions [11], as distinct from single-parameter families of nonlinear Schrödinger equation (NLSE) solitons [12]. Moreover, PCS behave like multiparticle objects in collisions [11].

Previous theoretical publications on incoherent solitons have dealt with stationary solutions [2,8], their internal dynamics [9,10], and their collisions [11,13]. The natural question now is how PCS interact with plane waves and, more specifically, whether a PCS can exist as an addition to a plane wave. In the case of a single NLSE, it is known that single solitons [12] and higher-order two-soliton solutions [14] can exist on a background. Interaction of the soliton with the plane wave causes nonlinear interference [15] and the net result is that the whole solution is periodic or quasiperiodic. We have found that analogous solutions exist for M coupled NLSEs. However, in the latter case, these solutions are stationary and do not change their shape on propagation, in contrast to the solutions of the single NLSE [12,14,15].

Incoherent self-trapping in a biased photorefractive crystal is usually well described by a set of M coupled NLSE equations with saturable nonlinearities [3]. Moreover, in photorefractive media with a drift mechanism of nonlinear response, the nonlinearity can be taken as Kerr-like [13]. Clearly, the Kerr-like nonlinearity is an approximation. Nevertheless, this approach provides us with valuable insight into the problem. In this case, the existence of exact solutions helps us to understand the phenomenon. The general idea is valid for any particular nonlinearity, so that solutions of our equations give a qualitative description of incoherent solitons in photorefractive media with more sophisticated nonlinearities.

In this work we present an exact solution to the coupled set of M equations which describes symmetric PCS on a background. Evidently, the restriction of symmetry and requirement of a sech profile reduces the multiparameter solutions to single parameter ones [13]. Here, the amplitude of the background gives the second parameter of the family, so that the solutions we are seeking are two-parameter families. Indeed, we find two-parameter families of PCS on a background for arbitrary integer M. In each case, one of the components has a nonzero asymptote, and this leads to a nonzero asymptotic value of the index change induced by the PCS. Note that these solutions cannot be obtained using the formalism of [16–18] because one of the functions does not decay to zero at infinity.

It can be shown [13] that, for photorefractive media with a drift mechanism of nonlinear response, a good approximation describing the propagation of M self-trapped mutually incoherent wave packets is the set of NLSE equations for a Kerr-type nonlinearity

$$i\,\frac{\partial \psi_j}{\partial z} + \frac{1}{2}\,\frac{\partial^2 \psi_j}{\partial x^2} + \alpha\,\delta n \psi_j = 0\,, \tag{1}$$

where ψ_j denotes the jth component of the beam, α is a coefficient representing the strength of nonlinearity, x is the transverse coordinate, z is the coordinate along the direction of propagation, and

$$\delta n = \sum_{j=1}^{M} |\psi_j|^2 \tag{2}$$

is the change in refractive index profile created by all the incoherent components in the light beam. Because the response time of the nonlinearity is assumed to be long compared to temporal variations of the relative phases of all the components, the medium responds to the average light intensity, and this is just a simple sum of modal intensities [19], as expressed by relation (2).

The set of equations (1) belongs to the class of integrable systems [20] and its solutions can be written in explicit forms. However, the inverse scattering technique in its standard formulation [17,20] and its equivalents [18] cannot be used here because of the nonzero boundary conditions. We use a direct method which gives the exact solution for arbitrary M and illustrate its application for $M = 3$ and $M = 4$. We are interested in stationary solutions of Eq. (1) which are given by

$$\psi_j(x,z) = \frac{1}{\sqrt{\alpha}}\, u_j(x) \exp(ik_j^2 z/2)\,, \tag{3}$$

with real functions $u_j(x)$, so that the set of Eqs. (1) reduces to the set of ordinary differential equations (ODEs):

$$\frac{d^2u_j}{dx^2} + 2\left[\sum_{i=1}^{M} u_i^2\right]u_j = k_j^2 u_j, \qquad j = 1,2,\ldots,M\,, \tag{4}$$

which is also completely integrable for an arbitrary set of real k_i. Various solutions to these equations, including soliton solutions [21–23] and periodic solutions [24,25] have been found, especially for $M = 2$ [26,27]. Examples of explicit solutions for $M > 2$ are still rare [13,24].

In particular, it is known [24] that for arbitrary k_1 and k_2, the set

$$\begin{aligned} u_1''(x) + 2(u_1^2 + u_2^2)u_1 &= k_1^2 u_1\,,\\ u_2''(x) + 2(u_1^2 + u_2^2)u_2 &= k_2^2 u_2\,, \end{aligned} \tag{5}$$

has a solution consisting of bright and dark soliton components. In fact, they are

$$\begin{aligned} u_1 &= \sqrt{a_2}\,\mathrm{sech}(\sqrt{f}\,x),\\ u_2 &= k_2 \tanh(\sqrt{f}\,x)/\sqrt{2}\,, \end{aligned} \tag{6}$$

where $f = k_1^2 - k_2^2$ and $a_2 = \frac{1}{2}(2k_1^2 - k_2^2)$ are constants. Interestingly enough, the total index change for this solution is

$$\delta n = u_1^2 + u_2^2 = k_2^2/2 + f\ \mathrm{sech}^2(\sqrt{f}\,x)\,, \tag{7}$$

where both the background and the central part of the solution are changed self-consistently by the components u_j. Hence this solution is a "soliton on a background." This simple example suggests that there could be higher-order solutions which have the same property. Below, we present these solutions in general form, for arbitrary integer M (>1).

Without loss of generality, we can arrange the eigenvalues in decreasing order $k_1 > k_2 > k_3 > \cdots > k_M$. Furthermore, we suppose that two of them, k_{M-1} and k_M, are arbitrary. We use a technique for obtaining the solution set based on the modes of the "sech-squared" waveguide [28]. The additional requirement here is that the change of the refractive index must be of the form const + $\mathrm{sech}^2(\sqrt{f}\,x)$, i.e., it should be similar to the function in Eq. (7). This means that each component must satisfy

$$\frac{d^2u_j}{dx^2} + [g_1 + g_2\,\mathrm{sech}^2(\sqrt{f}\,x)]u_j(x) = k_j^2 u_j(x)\,, \tag{8}$$

where $j = 1,2,\cdots M$, and g_1, g_2, and f are constants related to the k_j. We find that the solutions of (8) can be expressed as associated Legendre polynomials [29] starting from order 0; this distinguishes them from the solutions considered in [13] and [28] which were in terms of associated Legendre polynomials starting with order 1. We note that the well-known Legendre polynomials [29] $P_0 = 1, P_1 = \xi, P_2 = \frac{1}{2}(3\xi^2 - 1)$, etc., can be used to define the associated Legendre functions:

$$P_n^m(\xi) = (-1)^m(1 - \xi^2)^{m/2}\frac{d^m}{d\xi^m}P_n(\xi)\,. \tag{9}$$

We can rewrite (8) in the form

$$\frac{d^2u_j}{dx^2} + 2\,\delta n\, u_j(x) = k_j^2 u_j(x), \qquad j = 1,2,\cdots M\,. \tag{10}$$

such that

$$2\delta n = k_M^2 + fM(M-1)\,\mathrm{sech}^2(\sqrt{f}\,x)\,, \tag{11}$$

where δn is the total change of refractive index [see Eq. (2)]. By using $y = \sqrt{f}\,x$ and setting $f = k_{M-1}^2 - k_M^2$, where the eigenvalues k_{M-1} and k_M are arbitrary, each equation is transformed into

$$\frac{d^2u_j}{dy^2} + M(M-1)u_j\,\mathrm{sech}^2(y) = \lambda_j u_j\,, \tag{12}$$

where $\lambda_j = (k_j^2 - k_M^2)/(k_{M-1}^2 - k_M^2)$.

It is clear from Eq. (12) that $\lambda_M = 0$ and $\lambda_{M-1} = 1$. Hence, the two final components, u_M and u_{M-1}, have solutions in terms of the associated Legendre functions $P_{M-1}^0(\tanh y)$ and $P_{M-1}^1(\tanh y)$, respectively. To make other equations solvable in terms of the same functions, the coefficient λ_j on the right-hand side must be set to $(M-j)^2$. This means that we specify

$$k_j^2 = (M-j)^2(k_{M-1}^2 - k_M^2) + k_M^2\,, \tag{13}$$

so that, given the two eigenvalues, we can find the others directly from this relation, i.e., we can write down the whole set of k_j from the given values of k_{M-1} and k_M. Now, the component u_j appearing in Eq. (12) can be written in general form

$$u_j(y) = \pm c_{j,M}\,P_{M-1}^{M-j}(\tanh y), \quad j = 1,2,\ldots,M\,. \tag{14}$$

Hence the component u_j is an odd function of x if j is even, and is even if j is odd; it has $j-1$ zeros. The constant amplitudes $c_{j,M}$ must be chosen so that $\sum u_i^2 = \delta n$ always holds.

The following normalization can assist in finding these amplitudes:

$$\int_{-1}^{1} [P_M^j(\xi)]^2 \, d\xi = \frac{2}{2M+1} \frac{(M+j)!}{(M-j)!}. \qquad (15)$$

An important observation is that the Mth component u_M has a nonzero asymptote. This latter function can also be expressed in a different form. For a given M, it is clear that the final component, u_M, is proportional to P_{M-1}^0 and that this function has a nonzero value when $y \to \infty$, i.e., when $\xi = \tanh(y) \to 1$. All other components *do* approach zero in this limit. It is also obvious that, for a given M, each intensity, u_j^2, is a polynomial with terms in $\xi^0, \xi^2, \ldots, \xi^{2M-2}$, while the total intensity is a polynomial with constant and ξ^2 terms only. Setting $\xi = 1$ directly shows that

$$2u_M^2(\xi = 1) = k_M^2 \qquad (16)$$

since $u_j(\xi = 1) = 0$ if $j \neq M$. Hence the final amplitude constant can be found immediately.

Using the fact that $P_n(1) = 1$ for all n, we can further simplify the final component u_M in each case and write it in terms of a Legendre polynomial:

$$u_M(y) = \pm \frac{k_M}{\sqrt{2}} P_{M-1}(\xi), \qquad (17)$$

with $\xi = \tanh y$, as above. Clearly, the solution component u_2 in Eq. (6) belongs to this class.

Now we can write down the coefficients explicitly. By using $\xi = \tanh(y)$ in Eq. (11), integrating over $(-1, 1)$, and using Eqs. (14)–(17), we obtain

$$\sum_{j=1}^{M-1} \frac{(2M-j-1)!}{(j-1)!} (c_{j,M})^2 = \frac{(M-1)}{3} [M(2M-1)k_{M-1}^2 - (2M-3)(M+1)k_M^2]. \qquad (18)$$

We can satisfy this by choosing

$$c_{j,M} = \sqrt{\frac{(j-1)![2(M-j)^2(k_{M-1}^2 - k_M^2) + k_M^2]}{(2M-j-1)!}}, \qquad (19)$$

where $j = 1, 2, \ldots, M-1$. We recall also from Eqs. (14) and (17) that $c_{M,M} = k_M/\sqrt{2}$. With these values of the coefficients, Eq. (11) is satisfied exactly. Hence, the set of equations (13), (14), and (19) define the solution we are seeking for arbitrary M.

Let us present in explicit form two higher-order examples. For $M = 3$, we take k_2 and k_3 as arbitrary. Then $k_1^2 = 4k_2^2 - 3k_3^2$ and $f = k_2^2 - k_3^2$, and we obtain

$$\begin{aligned} u_1(x) &= \sqrt{a_3}\,\mathrm{sech}^2(\sqrt{f}\,x), \\ u_2(x) &= \sqrt{b_3}\,\mathrm{sech}(\sqrt{f}\,x)\tanh(\sqrt{f}\,x), \\ u_3(x) &= \sqrt{c_3}\,[1 - 3\tanh^2(\sqrt{f}\,x)]. \end{aligned} \qquad (20)$$

Here $a_3 = \frac{3}{8}(8k_2^2 - 7k_3^2)$, $b_3 = \frac{3}{2}(2k_2^2 - k_3^2)$, and $c_3 = \frac{1}{8}k_3^2$ are constants. The total index refractive change δn is given as

$$\sum_{i=1}^{3} u_i^2(x) = k_3^2/2 + 3f\,\mathrm{sech}^2(\sqrt{f}\,x). \qquad (21)$$

The three functions u_j given by (20) and the refractive index profile (21) (i.e., intensity), are shown in Fig. 1.

For $M = 4$, we take k_3 and k_4 as arbitrary. The solution set for $M = 4$ is

$$\begin{aligned} u_1(x) &= \sqrt{a_4}\,\mathrm{sech}^3(\sqrt{f}\,x), \\ u_2(x) &= \sqrt{b_4}\,\mathrm{sech}^2(\sqrt{f}\,x)\tanh(\sqrt{f}\,x), \\ u_3(x) &= \sqrt{c_4}\,\mathrm{sech}[(\sqrt{f}\,x)(5\tanh^2(\sqrt{f}\,x) - 1], \\ u_4(x) &= \sqrt{d_4}\,\tanh(\sqrt{f}\,x)[5\tanh^2(\sqrt{f}\,x) - 3]. \end{aligned} \qquad (22)$$

Here the constants are $a_4 = \frac{5}{16}(18k_3^2 - 17k_4^2)$, $b_4 = \frac{15}{8}(8k_3^2 - 7k_4^2)$, $c_4 = \frac{3}{16}(2k_3^2 - k_4^2)$, and $d_4 = \frac{1}{8}k_4^2$, while $f = k_3^2 - k_4^2$. The components u_j given by (22) are plotted in Fig. 2. The total index change δn is

$$\sum_{i=1}^{4} u_i^2(x) = k_4^2/2 + 6f\,\mathrm{sech}^2(\sqrt{f}\,x). \qquad (23)$$

The index change given by Eq. (23) has been omitted from Fig. 2 for clarity.

Now we show that the class of solutions with zero background given in [13] are special cases of the solutions which we have found here. If we specify the number of components, M, then we may choose the background intensity $(k_M^2/2)$ and the intensity at the point of maximum, $[k_M^2 + fM(M-1)]/2$, arbitrarily, since $f = k_{M-1}^2 - k_M^2$, and both k_M and k_{M-1} are arbitrary. Let us choose $k_M = 0$. Then we get zero background intensity, and one of the functions disappears, since $u_M = 0$. Hence, we obtain $M - 1$ nonzero components. In this case, $k_j/k_{M-1} = M - j$, $j = 1, 2, \ldots, M-2$, so

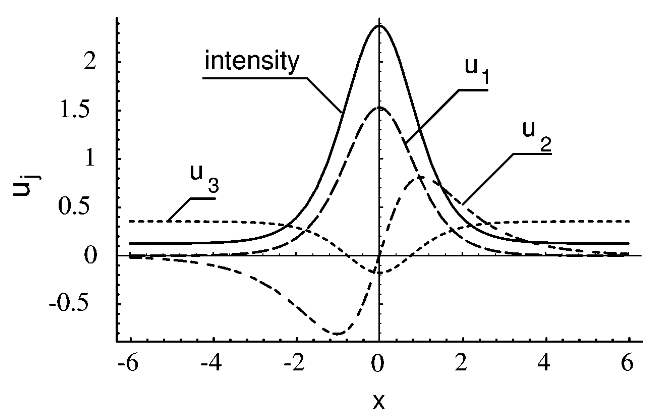

FIG. 1. The intensity profile of the soliton (solid line) and the amplitude profiles of all three linear modes u_1, u_2, and u_3 in the case $M = 3$. Parameters chosen in calculations are $k_2 = 1.0$ and $k_3 = 0.5$.

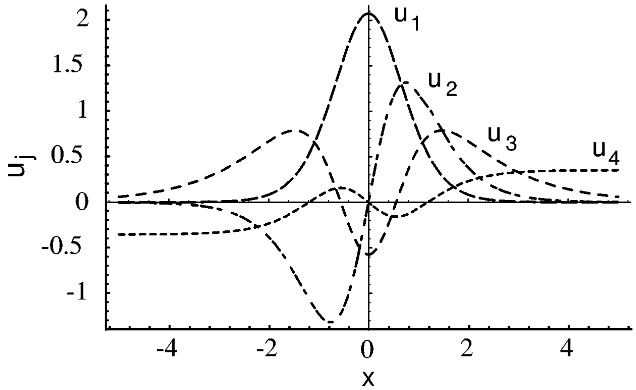

FIG. 2. The amplitude profiles of all four linear modes u_1, u_2, u_3, and u_4 in the case $M = 4$. Parameters chosen in calculations are $k_3 = 1.0$ and $k_4 = 0.5$.

the eigenvalues are all equally spaced. This corresponds to the special set (with $N = M - 1$) found in [13]. This explains why the solution set for equally spaced eigenvalues contains only one parameter, while our new solutions have 2 free parameters. For example, when $M = 4$, if we set $k_4 = 0$, then we have $d_4 = 0$, $c_4 = 3k_3^2/8$, etc., so $u_4 = 0$ while

$$u_3 = \frac{3k_3}{2\sqrt{6}}\,\mathrm{sech}(k_3 x)\,[5\tanh^2(k_3 x) - 1],$$

and similarly for other components. As a result, we obtain the set of zero background solutions with 3 components.

As the equations we are solving are integrable, it is clear that each of our solutions is a nonlinear superposition of a number of fundamental solitons with one finite amplitude plane wave. The first part of this superposition is neutrally stable, as for any other integrable system. The second part (the plane wave) is modulationally unstable [12]. This latter instability may cause the total solution to distort after some propagation distance. The instability growth rate depends on the plane wave amplitude and can be low. Then, significant changes will occur only on scales longer than the photorefractive crystal itself.

In conclusion, we have found the exact solution set for the symmetric sech-shaped PCS on a finite background. It can exist in photorefractive media with a drift mechanism of nonlinear response. The solution has two free parameters which are related to the maximum intensity of the soliton and the amplitude of the background.

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