

## Fermion-Boson Duality of One-Dimensional Quantum Particles with Generalized Contact Interactions

Taksu Cheon<sup>1,2</sup> and T. Shigehara<sup>3</sup>

<sup>1</sup>Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan

<sup>2</sup>Theory Division, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan

<sup>3</sup>Department of Information and Computer Sciences, Saitama University, Urawa, Saitama 338-8570, Japan

(Received 12 June 1998)

For a system of spinless one-dimensional fermions, the nonvanishing short-range limit of a two-body interaction is shown to induce the wave-function discontinuity. We prove the equivalence of this fermionic system and the bosonic particle system with a two-body  $\delta$ -function interaction with the reversed role of strong and weak couplings. [S0031-9007(99)08775-X]

PACS numbers: 71.10.Pm, 03.65.-w, 05.30.Fk

The relation between the spin and the exchange statistics is one of the fundamental properties of particles residing in four-dimensional Minkowski space. In lower dimension, however, the relation becomes blurred, as evidenced in the appearance of anyons in the system of two spatial dimension. In spatial dimension one, the relation loses its meaning since the spin itself is rather a phenomenological concept having no ground in the representation theory of a Lorentz group. The discovery of the strict equivalence of a bosonic sine-Gordon model and a fermionic massive Thirring model [1] suggests that the exchange statistics also is no absolute concept in one spatial dimension. Aside from its aesthetic value, this equivalence has practical ramifications in the treatment of an interacting many-body system in lower dimension. There the relevant aspect is the fact that the strong coupling in a fermionic model corresponds to the weak coupling in the bosonic model and vice versa.

There is indeed a historic precedence to the bosonization of fermionic theory in a setting of quantum many-body problem. In the Tomonaga-Luttinger theory of one-dimensional Fermi liquid [2–5], low energy excitations are describable in terms of bosonic degrees of freedom. Despite its status as a classical standard, the model has several drawbacks. First, the equivalence is exact only for the ground state of the system. Another problem is its nonapplicability to the short-range interaction as noted in the original paper by Tomonaga. This makes a sharp contrast to the case of bosons in one dimension where a simple but rich model of particles with two-body  $\delta$ -interaction exists [6], whose solvability allows the physical intuition as well as the thorough thermodynamical analysis.

The purpose of this paper is to formulate a model of a fermionic many-body system in one dimension with nonvanishing zero-range interaction. Its analysis reveals that the model can be exactly mapped to the same number of bosonic particles interacting through  $\delta$  interaction with the strength of the coupling reversed. This means that we have had a solvable model of interacting fermions for quite some time without recognizing it as such. It

gives a tractable model of a one-dimensional system with a nontrivial property of fermion-boson duality.

We start with a very elementary setting of two identical particles with unit mass in one dimension obeying the Fermi statistics. The wave function of the system has the property

$$\Psi_{-}(x_1, x_2) = -\Psi_{-}(x_2, x_1), \quad (1)$$

where  $x_1$  and  $x_2$  denote the coordinates of the particles. Let us suppose that the two particles are interacting through a two-body potential  $V(x_1 - x_2)$ . For now, we place one-body harmonic interaction for the technical convenience to bind the system around the origin. The Schrödinger equation is given by

$$\left[ \sum_{i=1}^2 \left( -\frac{1}{2} \frac{d^2}{dx_i^2} + \frac{1}{2} \omega^2 x_i^2 \right) + V(x_1 - x_2) \right] \Psi_{-}(x_1, x_2) = E \Psi_{-}(x_1, x_2). \quad (2)$$

With the usual use of the relative and center-of-mass coordinates  $x = x_2 - x_1$  and  $X = (x_1 + x_2)/2$ , the system separates into two subsystems as

$$\Psi_{-}(x_1, x_2) = \varphi_{-}(x) \Phi(X), \quad (3)$$

where the center-of-mass wave function  $\Phi(X)$  is a trivial harmonic oscillator, and the physics is in the relative wave function  $\varphi_{-}(x)$ , which satisfies

$$\left[ -\frac{d^2}{dx^2} + \frac{1}{4} \omega^2 x^2 + V(x) \right] \varphi_{-}(x) = E^r \varphi_{-}(x). \quad (4)$$

The identity of the particles requires  $V$  to be symmetric:

$$V(-x) = V(x). \quad (5)$$

The fermionic exchange symmetry, Eq. (1), now reads

$$\varphi_{-}(-x) = -\varphi_{-}(x). \quad (6)$$

We consider the case where the potential is short ranged, namely,  $V(x) = 0$  for  $|x| > a$  for a small positive number  $a$ . At the limit  $a \rightarrow 0$ , the self-adjoint extension theory dictates that any Hermitian potential has to be reduced to the generalized pointlike interaction [7–10]

$$V(x) \rightarrow \chi(x; \alpha, \beta, \gamma, \delta), \quad (7)$$

which admits the discontinuity both of the wave function and its space derivative. The physical understanding of this rather counterintuitive object  $\chi$  is possible through its explicit construction in terms of a local operator, which has recently been devised [11]. We have

$$\chi(x; \alpha, \beta, \gamma, \delta) = \lim_{a \rightarrow +0} [u_- \delta(x + a) + u_0 \delta(x) + u_+ \delta(x - a)], \quad (8)$$

where the strengths of  $\delta$  functions are given by

$$\begin{aligned} u_+(a) &= -\frac{1}{a} + \frac{\alpha - 1}{\delta}, \\ u_-(a) &= -\frac{1}{a} + \frac{\gamma - 1}{\delta}, \\ u_0(a) &= \frac{1 - \alpha\gamma}{\beta a^2}, \end{aligned} \quad (9)$$

in which  $\alpha, \beta, \gamma,$  and  $\delta$  are arbitrary real numbers with the constraint

$$\alpha\gamma - \beta\delta = 1. \quad (10)$$

The effect of the  $\chi(x)$  on the wave function can be expressed as

$$\begin{aligned} \varphi'_-(0_+) + \alpha\varphi'_-(0_-) &= -\beta\varphi_-(0_-), \\ \varphi_-(0_+) + \gamma\varphi_-(0_-) &= -\delta\varphi'_-(0_-), \end{aligned} \quad (11)$$

where prime signifies the spatial derivative. In the current case, these conditions have to be compatible with the antisymmetry of wave functions, Eq. (6), which results in  $\varphi_-(0_+) = -\varphi_-(0_-)$  and  $\varphi'_-(0_+) = \varphi'_-(0_-)$ . Also, from the symmetry of  $V(x)$ , Eq. (5), one has  $u_+(a) = u_-(a)$ , thus  $\alpha = \gamma$ . The condition, Eq. (11), is then reduced to

$$\varphi_-(0_+) = -\varphi_-(0_-) = 2c\varphi'_-(0_+) = 2c\varphi'_-(0_-), \quad (12)$$

where the real number  $c$  is defined by

$$c = \frac{\delta}{2(\alpha - 1)}. \quad (13)$$

We therefore have the discontinuity in the *wave function itself*, while its derivative is kept continuous. The short-range limit, Eq. (7), now reads

$$V(x)\varphi_-(x) \rightarrow \varepsilon(x; c)\varphi_-(x) \quad (a \rightarrow 0), \quad (14)$$

where  $\varepsilon(x; c)$  is defined by

$$\begin{aligned} \varepsilon(x; c)\varphi_-(x) &= \lim_{a \rightarrow +0} \left( \frac{1}{2c} - \frac{1}{a} \right) \\ &\times \{ \delta(x + a) + \delta(x - a) \} \varphi_-(x). \end{aligned} \quad (15)$$

The easiest (but not unique) way to express  $\varepsilon(x)$  in terms of  $\chi(x)$  is

$$\varepsilon(x; c) = \chi(x; -1, 0, -1, -4c). \quad (16)$$

This should be contrasted to the ‘‘usual’’ zero-range limit, Dirac’s delta function

$$\delta(x; \nu) \equiv \chi(x; -1, -\nu, -1, 0) = \nu\delta(x), \quad (17)$$

which has no effect on the antisymmetric wave function;

$$\delta(x; \nu)\varphi_-(x) = 0. \quad (18)$$

Thus the nonvanishing zero-range limit of the system is described by

$$\left[ -\frac{d^2}{dx^2} + \frac{1}{4}\omega^2x^2 + \varepsilon(x; c) \right] \varphi_-(x) = E^r \varphi_-(x). \quad (19)$$

Intuitive meaning of the admissibility of the discontinuity-inducing interaction  $\varepsilon(x; c)$  should become clear by inspecting Fig. 1, where we depict the antisymmetric wave functions subjected to a symmetric potential of small but finite range. The procedure, Eqs. (14) and (15), is a nontrivial but sensible renormalized zero-range limit that keeps the nonvanishing effect of the potential through a rather unfamiliar concept of wave-function discontinuity. Further analysis of the renormalization properties of  $\varepsilon(x; c)$  should be of special interest, since this has the direct relevance to the question of approximating the more realistic, finite-range problem with proper choice of the coupling  $c$ .

We now perform a transformation:

$$\varphi_+(x) = [\theta(x) - \theta(-x)]\varphi_-(x), \quad (20)$$

where  $\theta(x)$  is the step function  $\theta(x) = 1$  when  $x > 0$  and  $\theta(x) = 0$  when  $x < 0$ . The connection condition, Eq. (12), is rewritten as

$$\varphi'_+(0_+) = -\varphi'_+(0_-) = \frac{1}{2c}\varphi_+(0_+) = \frac{1}{2c}\varphi_+(0_-), \quad (21)$$

which means that  $\varphi_+(x)$  satisfies an analog of Eq. (11) with  $\alpha = \gamma = -1$ ,  $\delta = 0$ , and  $\beta = -1/c$ . In other words,  $\varphi_+(x)$  is a solution of the Schrödinger equation,

$$\left[ -\frac{d^2}{dx^2} + \frac{1}{4}\omega^2x^2 + \delta(x; \nu) \right] \varphi_+(x) = E^r \varphi_+(x), \quad (22)$$

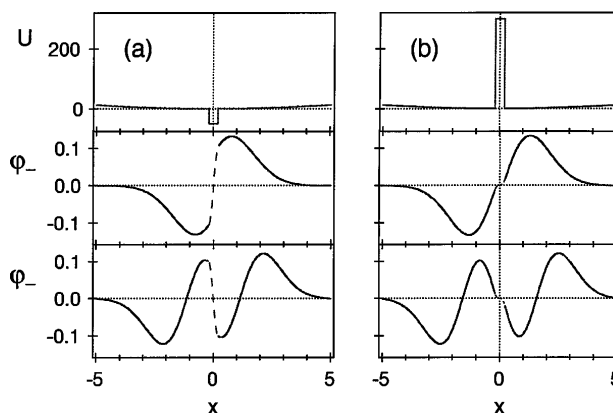


FIG. 1. Examples of fermionic relative wave functions. The harmonic oscillator parameter  $\omega = 2$  sets the scale. The interaction  $V$  is chosen to be a square well with the range  $a = 0.2$ . (a) The case for attractive interaction of depth  $-50$ . (b) The case of repulsive interaction of height  $300$ . In both cases, top figures are the profile of the interaction  $U = \omega^2x^2/4 + V$ . The middle and the bottom are the lowest and the second lowest energy eigenfunctions.

if the coupling constants  $v$  and  $c$  are related by

$$v = \frac{1}{c}. \quad (23)$$

By construction, one has

$$\varphi_+(-x) = \varphi_+(x). \quad (24)$$

In terms of the full two-particle wave function

$$\Psi_+(x_1, x_2) = \varphi_+(x)\Phi(X), \quad (25)$$

this signifies the bosonic exchange symmetry

$$\Psi_+(x_1, x_2) = \Psi_+(x_2, x_1). \quad (26)$$

Therefore, a two-fermion system with  $\varepsilon$  interaction is equivalent to a two-boson system with  $\delta$  interaction, and the strong coupling in one side corresponds to the weak coupling in the other. We emphasize that  $\delta$  and  $\varepsilon$  functions are the *only* nonvanishing limits of any interaction that acts on bosonic and fermionic wave functions, respectively. Note the parallel relation to Eq. (18) for  $\varphi_+(x)$ ;

$$\varepsilon(x; c)\varphi_+(x) = 0. \quad (27)$$

Note also that the couplings  $v$  and  $c$  can be both positive and negative. In the latter case, the equivalence extends to the negative-energy bound states that exist in both Fermi and Bose systems.

It is instructive to look at the wave functions to see the actual workings of the boson-fermion duality with some numerical examples. We show, in Fig. 2(a), the lowest energy fermionic eigenstates of Eq. (19) with several values of coupling strengths. In Fig. 2(b), the corresponding bosonic eigenstates of Eq. (22) are displayed. In the calculations, the  $\varepsilon$  interaction, Eq. (15), is evaluated with a small but finite value of  $a$  in place of the  $a \rightarrow 0$  limit. These figures show that the rigorous results at the mathematical limit  $a \rightarrow 0$  do have real relevance to a more realistic problem with finite-range interactions.

It is straightforward to extend the above arguments to the system of  $N$  one-dimensional particles. Let us write the wave function of the system for the particular ordering of the set of  $N$  coordinates  $(x_1, x_2, \dots, x_N)$ , say  $x_1 > x_2 > \dots > x_N$ , as  $\Psi_1$ ;

$$\Psi_1 \equiv \Psi(x_1, \dots, x_N)\theta(x_1 - x_2)\cdots\theta(x_{N-1} - x_N). \quad (28)$$

We define the permutation  $P$  of  $N$  numbers:

$$P: (1, 2, \dots, N) \rightarrow (P_1, P_2, \dots, P_N). \quad (29)$$

$$\left[ \sum_i \left( -\frac{1}{2} \frac{d^2}{dx_i^2} + \frac{1}{2} \omega^2 x_i^2 \right) + \sum_{i>j} \varepsilon(x_i - x_j; c) \right] \Psi_-(x_1, \dots, x_N) = E \Psi_-(x_1, \dots, x_N) \quad (34)$$

and

$$\left[ \sum_i \left( -\frac{1}{2} \frac{d^2}{dx_i^2} + \frac{1}{2} \omega^2 x_i^2 \right) + \sum_{i>j} \delta\left(x_i - x_j; \frac{1}{c}\right) \right] \Psi_+(x_1, \dots, x_N) = E \Psi_+(x_1, \dots, x_N), \quad (35)$$

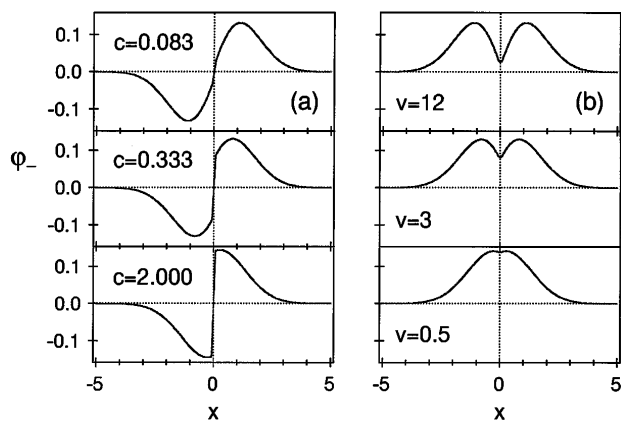


FIG. 2. The fermionic (a) and bosonic (b) relative wave functions with three values of coupling parameter. Wave functions (a) and (b) are related by the transformation, Eq. (20). The  $\varepsilon$  interaction for case (a) are constructed from Eq. (15) with the  $a \rightarrow 0$  limit replaced by a small number  $a = 0.05$ .

Suppose  $(-1)^P$  represents the parity of the permutation  $P$ . The wave functions  $\Psi_{\pm}$  defined by

$$\Psi_{\pm}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \sum_P (\pm 1)^P \Psi_1(x_{P_1}, \dots, x_{P_N}) \quad (30)$$

have the exchange symmetry

$$\Psi_{\pm}(\dots, x_i, \dots, x_j, \dots) = \pm \Psi_{\pm}(\dots, x_j, \dots, x_i, \dots). \quad (31)$$

Namely,  $\Psi_+$  and  $\Psi_-$  represent the systems of  $N$  bosons and  $N$  fermions, respectively. It is easy to see that the following two equations are equivalent:

$$\begin{aligned} \Psi_-|_{x_i=x_{j+}} &= -\Psi_-|_{x_i=x_{j-}} \\ &= c \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Psi_- \Big|_{x_i=x_{j+}} \\ &= c \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Psi_- \Big|_{x_i=x_{j-}}, \quad (32) \\ \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Psi_+ \Big|_{x_i=x_{j+}} &= - \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \Psi_+ \Big|_{x_i=x_{j-}} \\ &= \frac{1}{c} \Psi_+|_{x_i=x_{j+}} = \frac{1}{c} \Psi_+|_{x_i=x_{j-}}. \quad (33) \end{aligned}$$

Therefore,  $\varepsilon(x_i - x_j; c)$  acting on  $\Psi_-$  and  $\delta(x_i - x_j; 1/c)$  acting on  $\Psi_+$  are two different representations of the same effect. We have the equivalence of two equations,

that can be mapped into each other.

The confining harmonic potential is an artifact to supply the basis functions, which sometimes causes a nuisance. Alternatively, one sets  $\omega = 0$  and imposes the cyclic boundary condition:

$$\Psi(\dots, x_i + L, \dots) = \Psi(\dots, x_i, \dots) \quad \text{for } i = 1, \dots, N. \quad (36)$$

There is a subtle complication with this prescription [6,12], which we analyze in the following. Let us suppose, for a moment, that we have a set of  $x_i$  all within the range of length  $L$ , say  $\frac{L}{2} > x_i > -\frac{L}{2}$  with the ordering  $x_1 > x_2 > \dots > x_N$ . By definition, one has  $\Psi_{\pm}(x_1, \dots, x_N) = \Psi_1(x_1, \dots, x_N)$ . With the replacement  $x_N \rightarrow x_N + L$ , one has  $\Psi_{\pm}(x_1, \dots, x_N + L) = (\pm 1)^{N-1} \Psi_1(x_N + L, x_1, \dots, x_{N-1})$ . This can be rewritten as a relation between  $\Psi_+$  and  $\Psi_-$  in the form  $\Psi_-(x_1, \dots, x_N + L) = (-1)^{N-1} \Psi_+(x_1, \dots, x_N + L)$ . Thus it is not always appropriate to impose the cyclic boundary both for  $\Psi_+$  and  $\Psi_-$ . A consistent description of the boundary is achieved by replacing the strict periodic condition, Eq. (36), with a relaxed version

$$\Psi_{\pm}(\dots, x_i + L, \dots) = e^{i\lambda_{\pm}} \Psi_{\pm}(\dots, x_i, \dots) \quad (37)$$

for  $i = 1, \dots, N$

with

$$\lambda_- = \lambda_+ + (N - 1)\pi.$$

Then, for  $\omega = 0$ , the fermionic problem, Eq. (34), is equivalent to the bosonic problem, Eq. (35). Specifically, the usual choice  $\lambda_+ = 0$  gives the periodic boundary for  $\Psi_+$  and antiperiodic boundary for  $\Psi_-$ .

The representation of our model is the second-quantized form should be very useful, since it is in that form that the bosonization of fermion systems is discussed with formidable mathematical machinery [13,14]. Also, it could lead to a new type of field theoretical model. A technical block on its way is the nonperturbative nature of the  $\varepsilon$  interaction, which does not allow meaningful calculations of its matrix elements in a straightforward manner.

Since the complete solution based on the Bethe ansatz for the bosonic problem, Eq. (35), with  $\omega = 0$  exists [6], we now have a model of a solvable fermion  $N$ -body problem with nontrivial characteristics. It is of particular interest to investigate the thermodynamic properties of this system in detail.

It would be worthwhile to place our approach in the context of other solvable many-body models in one dimension [15]. In particular, the study of its relation (or contrast) to the model with a long-range interaction,

namely, the Calogero-Sutherland model [16–18], appears to be a promising subject. It should be also interesting to look at the fermion-boson relations in *other* dimensions. This is especially true in light of a recent article on the equivalence between free fermions and free bosons in dimension two [19]. Finally, we call the reader's attention to rather unexplored potential roles of the generalized contact interactions in other contexts than discussed here. Those include such diverse subjects as the semiconductor heterojunctions [20] and the controversy over the one-dimensional  $1/|x|$  potential [21,22].

We are very grateful to Professor T. Tsutsui and Dr. T. Fülöp for valuable comments. Thanks are also due to Professor Y. Okada and the members of the Theory Group for the hospitality extended to T.C. at KEK, Tsukuba. This work has been supported in part by the Grant-in-Aid (No. 10640396) by the Japanese Ministry of Education.

- 
- [1] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
  - [2] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
  - [3] J. M. Luttinger, J. Math. Phys. (N.Y.) **4**, 1154 (1963).
  - [4] D. C. Mattis and E. Lieb, J. Math. Phys. (N.Y.) **6**, 304 (1965).
  - [5] F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).
  - [6] E. Lieb and W. Linger, Phys. Rev. **130**, 1605 (1963); **130**, 1616 (1963).
  - [7] F. Gesztesy and W. Kirsch, J. Reine Angew. Math. **362**, 28 (1985).
  - [8] P. Šeba, Czech. J. Phys. B **36**, 667 (1986).
  - [9] P. Šeba, Rep. Math. Phys. **24**, 111 (1986).
  - [10] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, Heidelberg, 1988).
  - [11] T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998).
  - [12] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).
  - [13] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, England, 1987).
  - [14] P. Kopietz, *Bosonization of Interacting Fermions in Arbitrary Dimensions* (Springer-Verlag, Heidelberg, 1997).
  - [15] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94**, 313 (1983).
  - [16] B. Sutherland, J. Math. Phys. **12**, 246 (1971); **12**, 251 (1971).
  - [17] F. Calogero, J. Math. Phys. **12**, 419 (1971).
  - [18] Z. N. C. Ha, *Quantum Many-Body Systems in One Dimension* (World Scientific, Singapore, 1996).
  - [19] M. H. Lee, Phys. Rev. E **55**, 1518 (1997).
  - [20] R. Balian, D. Bessis, and G. A. Mezincescu, Phys. Rev. B **51**, 17 624 (1995).
  - [21] R. Loudon, Am. J. Phys. **27**, 649 (1959).
  - [22] A. N. Gordeyev and S. C. Chhajlany, J. Phys. A **30**, 6893 (1997).