Control of Unstable High-Period Orbits in Complex Systems

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We apply the discrete parametric control (Ott-Brebogi-York method) to stabilize the high-order ($k = 34$) unstable periodic orbit of a 2D chaotic map. The map describes a complex reversible system, the phase space of which contains elements typical for both Hamiltonian and dissipative dynamics. Stabilization was achieved (even with external noise) for the unstable orbit where the amplitude of chaotic oscillations shows variations by 4 orders of magnitude. [S0031-9007(99)08706-2]

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The recent tendency in the area of controlling chaos is related to control of complex systems in order to carry out, possibly, control of biological and even human functions. Despite the great variety in the properties of complex systems, it is assumed, nevertheless, that any of these systems possess certain common features: (i) A complex system is a composition of several interacting components; (ii) there is a coexistence of regular and chaotic dynamics; and (iii) it exhibits a multiscale spatiotemporal behavior [1]. These features make it possible to control complex system, using a rather simple switching of a system from chaotic to regular regime (or vice versa) under a small perturbation.

The above mentioned complexity is not necessarily an attribute of high-dimensional systems only; it may also be found in low-dimensional systems. In particular, the low-dimensional dynamics is realized in so-called reversible systems which exhibit typical complex behavior [2]. The phase space of such systems usually contains elements of Hamiltonian systems (stability islands and resonances) as well as elements of dissipative systems (attractors and invariant attractive sets) [3]. The interplay between these elements gives rise to rather complicated dynamics as compared with the dynamics of either pure Hamiltonian or pure dissipative systems.

In this Letter, we accomplish, for the first time, the control of chaos in complex reversible systems. We study the possibility of controlling a high-period ($k = 34$) unstable periodic orbit (UPO) in a two-dimensional map:

$$\mathbf{r}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{f}(\mathbf{r}_n) = \begin{pmatrix} x_n + y_{n+1} \ (\mathrm{mod}2) \\ y_n - \varepsilon(a - y_n)x_n \end{pmatrix}. \quad (1)$$

Equation (1) describes the discrete dynamics of a linear oscillator with the stiffness coefficient proportional to the velocity, under the action of δ kiks [4]. The phase space of this map is a cylinder, $x \in [-1, 1]$, $y \in R$. For $\varepsilon > 0$, the phase space structure is determined by the invariant attractive set (IAS) at $y = a$ and fixed points at $(0, 2n)$, $n = 0, \pm 1, \pm 2$. The linear analysis shows that the fixed points within the stripe $a - 4/3 < y < a$ are elliptic centers and the others are hyperbolic saddles. IAS is formed from the set of solutions,

$$y_n = a, \qquad x_{n+1} = (x_n + y_{n+1}) \qquad (\mathrm{mod}2), \qquad (2)$$

which are periodic (quasiperiodic) for rational (irrational) values of the parameter a. It was shown [5] that the set [Eq. (2)] attracts trajectories (belonging to the attractor's basin) with increment $\gamma \cong \varepsilon^2/6$ ($\varepsilon < 1$). The number of trajectories that are repelled from the solution [Eq. (2)] and approach infinity (along the separatrices of saddle points) increases with ε. Thus we have two attractive regions, one given by Eq. (2) and another located at the infinity. The basins of these two attractive sets have a complicated fractal structure which is typical for any riddled basin [6].

Hamiltonian properties of the map [Eq. (1)] are determined by the determinant of the Jacobian matrix $\det \mathbf{A} = 1 + \varepsilon x$ which gives the variation of the phase volume after one iteration. At each iteration the phase volume is not preserved; however, there is a conservation of volume after averaging over the period. For $x \ll 1$, the determinant of the Jacobian matrix is close to 1; therefore the structure of the phase space in the vicinity of the origin is similar to the one in a Hamiltonian system: Namely, the elliptic fixed point (0,0) is surrounded by the periodic trajectories. The frequency of rotation for periodical trajectories close to an elliptic point is $\omega = 2 \arcsin(\sqrt{\varepsilon a}/2)$. A fragment of the phase space (stability island) is shown in Fig. 1a for the values of the parameters $a = 0.5, \varepsilon = 1.7$. Deformation of this stability island with parameter a is shown in Fig. 1b for $a = 0.05$. One can see that with the decreasing of a the stability island and the IAS are getting closer and the separatrices of high-order resonances are being broken. A narrow stochastic layer is formed from these broken separatrices (see Fig 1c). One may conclude that the origin of chaos here is due to the overlapping of neighboring nonlinear resonances. Figure 2 shows a plot of the width W of the resonances and the spacing Δ between them against k. For $k > 30$, the width W and the spacing Δy are of the same order of magnitude and at $k \approx 33$–35

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a global chaos is developed, according to Chirikov's criterion [7]. Here we choose the UPO with $k = 34$ which we are going to control.

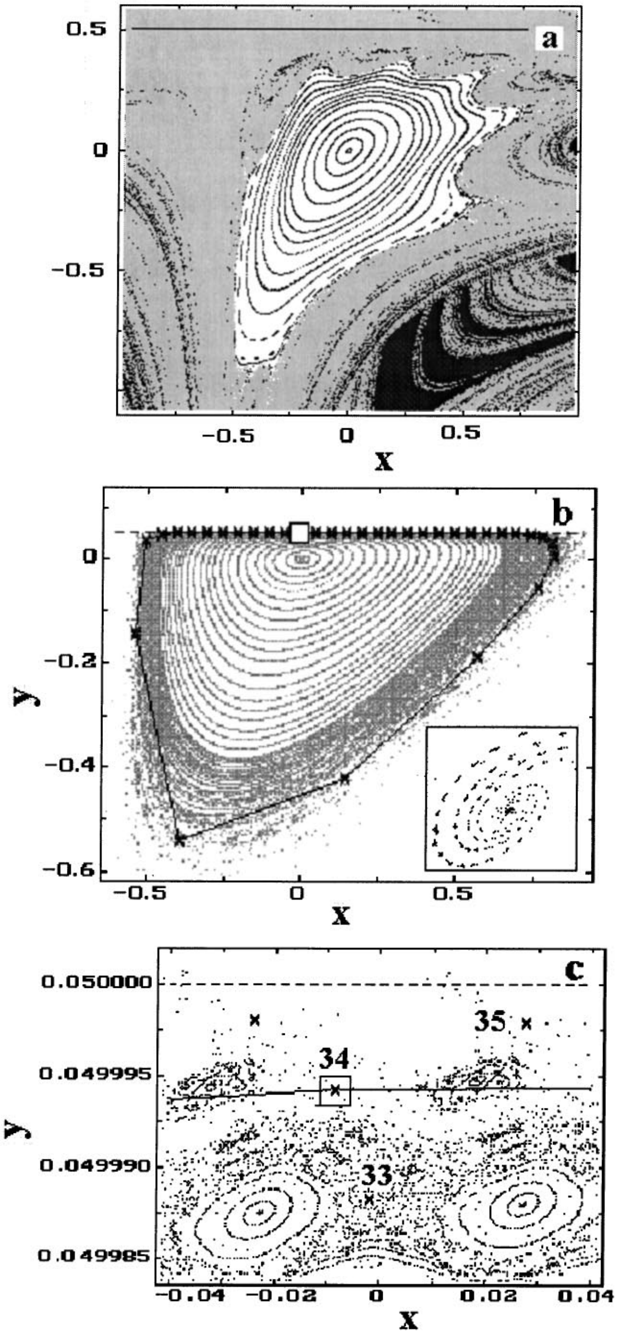

FIG. 1. (a) Fragment of the phase space of the map [Eq. (1)] with stability island at the center ($a = 0.5, \varepsilon = 1.7$). (b) Deformation of the stability island with parameter a: $a = 0.05, \varepsilon = 1.7$. Period-34 orbit is marked by crosses connected by straight lines. (c) Blow up of a small region of the phase space [small blank square in (b)]. One of the saddle points of the orbit is marked by number 34. Kolmogorov-Arnold-Moser surfaces close to the period-35 orbit are destroyed, however, they still exist for period-33 orbit. Inset of (b): image of the saddle point [marked by number 34 in (c)] after transformation Eq. (3).

The first question which arises is how to locate this UPO. Traditional methods of locating UPOs based on the Newton-Raphson procedure require a good guess of initial conditions for the iterative procedure. In general, they are not applicable for cycles with high periods. An appropriate method was developed by Schmelcher and Diakonos [8]. They used the principal idea of control—transforming unstable orbits into stable ones—to locate UPOs. The first step of the method [8] is to apply a universal linear transformation of coordinates in order to get stable orbits at the same positions where unstable orbits are located. Then the positions of stable orbits in new coordinates can be found by a simple iterative procedure. For the 2D case the transformation of coordinates takes the following form:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{\Lambda}_i[\mathbf{f}^m(\mathbf{r}_n) - \mathbf{r}_n], \tag{3}$$

where $\mathbf{\Lambda}_i$ is one of eight ($i = 1, 2, ..., 8$) invertible 2×2 matrices (for D-dimensional case $i = D!2^D$), the concrete form of which is determined by type of the orbit. The inset in Fig. 1b shows the result of transformation [Eq. (3)] for one of the saddle points lying on the period-34 orbit.

To stabilize the high-period unstable orbit, we used the discrete one-parametric Ott-Grebogi-York (OGY) control [9]. This method was originally proposed to stabilize UPO embedded within a strange attractor. Later it was generalized for the case of Hamiltonian systems [10]. The orbit to be controlled follows a periodic sequence, $\mathbf{r}_1^* \to \mathbf{r}_2^* \to \cdots \to \mathbf{r}_k^* \to \mathbf{r}_{k+1}^* = \mathbf{r}_1^*$. The linearized dynamics in the neighborhood of this orbit yields

$$\mathbf{r}_{n+1} - \mathbf{r}_{n+1}^*(p_0) = \mathbf{A}[\mathbf{r}_n - \mathbf{r}_n^*(p_0)] + \mathbf{B}\delta p_n, \tag{4}$$

where p is one of the parameters a, ε; p_0 is its nominal value; $p_n - p_0 \equiv \delta p_n < \Delta$; Δ is the range of variations of the parameter p ($\Delta \ll 1$); $\mathbf{A}$ is 2×2 Jacobian matrix; and $\mathbf{B}$ is two-dimensional vector,

$$\mathbf{A} = D_r\mathbf{f}(\mathbf{r}, p)|_{\substack{r=r_n^* \\ p=p_0}}, \qquad \mathbf{B} = D_p\mathbf{f}(\mathbf{r}, p)|_{\substack{r=r_n^* \\ p=p_0}}. \tag{5}$$

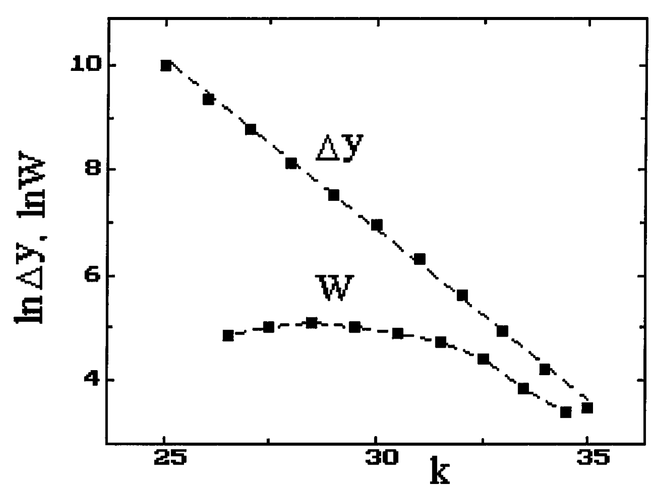

FIG. 2. The width of the resonance (W) and the spacing between them (Δy) vs the number of the resonance (k).

If we try to apply the OGY method [9] directly to control high-period unstable orbits, we face two difficulties. First, the one-step Jacobian matrix **A** may possess complex eigenvalues at certain points of a periodic orbit. This makes the application of the original OGY method impossible. To avoid this, we could use the original OGY formula, not at each iteration, but after k iterations. In other words we could perform control with a cyclic matrix $\mathbf{f}^k$ whose eigenvalues are real. However, here we face the second difficulty, namely, the sensitivity of k-step control to external or system (numerical) noise. This means that k-step control becomes ineffective if $k \gg 1$. A modification of the OGY method which allows the application of parametric perturbation at each step was proposed by Lai *et al.* [10]. To avoid the problem of complex eigenvalues they developed the approach based on the consideration of stable and unstable directions at each point of the UPO. If $k > 1$, these directions (at a given point) do not necessarily coincide with the eigenvectors. An efficient method to calculate stable and unstable directions is given in [10]. They can be also calculated by the well-known method of diagonalization of the cyclic matrix $\mathbf{f}^k$ at each point of the orbit.

Let the unit vectors $\mathbf{e}_{s(n)}$ and $\mathbf{e}_{u(n)}$ be local ($\mathbf{r} = \mathbf{r}_n^*$) stable and unstable directions. It is worthwhile to introduce a complementary basis $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ by means of the following relations: $\mathbf{f}_{s(n)}^t \mathbf{e}_{s(n)} = \mathbf{f}_{u(n)}^t \mathbf{e}_{u(n)} = 1$, and $\mathbf{f}_{u(n)}^t \mathbf{e}_{s(n)} = \mathbf{f}_{s(n)}^t \mathbf{e}_{u(n)} = 0$. Here index t stands for transposed vector (row vector). By making use of the OGY stabilization condition $\mathbf{f}_{u(n+1)}^t[\mathbf{r}_{n+1} - \mathbf{r}_{n+1}^*(p_0)] = 0$ and Eq. (4), one can readily get [11]

$$\delta p_n = -\frac{\mathbf{f}_{u(n+1)}^t \{\mathbf{A}[\mathbf{r}_n - \mathbf{r}_n^*(p_0)]\}}{\mathbf{f}_{u(n+1)}^t \mathbf{B}}. \qquad (6)$$

Parametric perturbation written in the form of Eq. (6) may be applied at each iteration. If the eigenvalues of the matrix **f** are real, Eq. (6) is reduced to the OGY formula [9].

Figure 3 demonstrates the internal mechanism of OGY control in action. We "launch" four testing points (black squares) from the vicinity of randomly selected saddle points which belong to period-34 orbit. The trajectories of the testing points are shown after three successive iterations. After the third iteration these points are aligned along the stable direction. Then they follow the periodic orbit remaining in alignment and approaching the saddle points with each iteration.

In Fig. 4a we show the behavior of the deviation $\mathbf{r}_n - \mathbf{r}_n^*$ as the control is switched on and off. One can see that the system exhibits a long transient period before a trajectory can be stabilized. Control is switched on at the 340th iteration; however, stabilization occurs only after approximately 1000 iterations. Such behavior is similar to what has been observed for pure Hamiltonian systems [10,12]. We use the logarithmic scale to separate different stages of the control procedure: (i) chaotic oscillations before entrapment under control; (ii) exponentially fast approach of the controlled trajectory to the goal-periodic orbit; (iii) *stable* motion along *unstable* period-34 orbit (up to the numerical error accuracy); (iv) exponential deviation from the goal trajectory after the control is off [13], (v) reconstruction of natural chaotic oscillations.

To check the sensitivity of the control to external Gaussian noise we add a term $s\boldsymbol{\xi}_n$ to the right-hand side in Eq. (1), Here ξ_{xn} and ξ_{yn} are independent identically distributed random variables with zero mean value and a unit variance; s is the amplitude of the noise. In Fig. 4b we show the influence of the Gaussian noise with $s = 0.01$. In the logarithmic scale, one can clearly see that the efficiency of control goes down by the orders of magnitude; however, the OGY method allows the control to be maintained during the same temporal interval as without noise.

In addition to the OGY method, we also applied the simple proportional method [14]. Originally it was proposed for the control of Hamiltonian systems [15]. In this method a perturbation $\boldsymbol{\Delta}_n$ stands in the right-hand side of the map [Eq. (1)] and has a form $\boldsymbol{\Delta}_n = \mathbf{C}(\mathbf{r}_n - \mathbf{r}_n^*)$, where the matrix $\mathbf{C} = \begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix}$. The perturbation $\boldsymbol{\Delta}_n$ is applied if $\Delta_n < \Delta_{\max}$ ($\Delta_{\max} \ll 1$). By choosing parameters $c_1 = -0.2, c_2 = 0.0001$ and $\Delta_{\max} = 0.02$ we were able to control the period-34 orbit with the same accuracy as that shown in Fig. 3 for the OGY method.

The proposed method of control may be used for a wide class of reversible systems with complex phase space structure. In particular, a simple modification of the map (1) allows one to change the level of dissipation. This is done by introducing a coefficient $1 - c$ for the y_n term in the second equation of the map (1). When the dissipation is switched on smoothly (c increases from 0 to 1), the Hamiltonian component is suppressed and the system developes a strange attractor via a cascade of bifurcations. The results of the control for the case when $c \neq 0$ will be published elsewhere.

An interesting application of controlling chaos is related to the problem of encoding of binary information

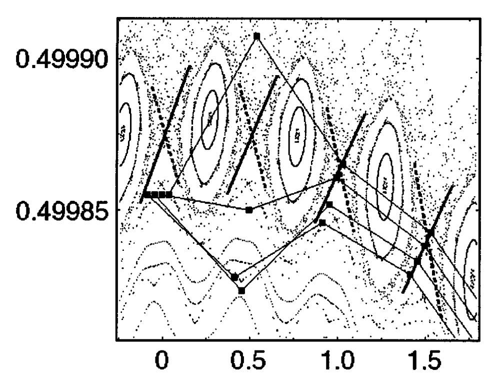

FIG. 3. Local evolution of four testing points towards the stable direction. Stable (unstable) directions are shown by thick solid (broken) lines.

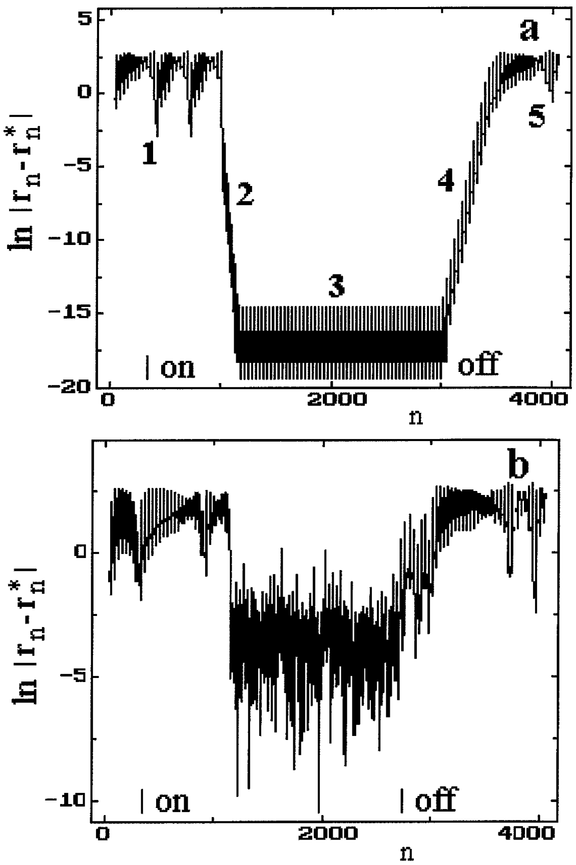

FIG. 4. Stabilization of the coordinate $\mathbf{r}_n$ when the control is switched on: (a) without noise, (b) with Gaussian noise.

[16]. Stabilized orbits serve as generators of random numerical sequences that are considered as carriers of information. High-period orbits give a rich grammar (the rules specifying allowed and disallowed symbol sequences) and high density of information in the sequence. The high density of information, plus the stability with respect to noise, gives our method great potential for applications in communication.

In conclusion, we have demonstrated the efficiency of the discrete parametric OGY control for UPO ($k = 34$) in a complex reversible system with attributes of Hamiltonian and dissipative dynamics. To the best of our knowledge, this is the first case of control of such a high-order UPO in complex systems. The problem of location of the period-34 orbit was resolved by making use of a new method proposed in [8]. Stabilization was successfully achieved when the amplitude of chaotic oscillations of coordinates along the period-34 orbit was varied by 4 orders of magnitude.

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