# Collective Motion of Self-Propelled Particles: Kinetic Phase Transition in One Dimension

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We demonstrate that a system of self-propelled particles exhibits spontaneous symmetry breaking and self-organization in one dimension, in contrast with previous analytical predictions. To explain this surprising result we derive a new continuum theory that can account for the development of the symmetry broken state and belongs to the same universality class as the discrete self-propelled particle model. [S0031-9007(98)07911-3]

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The transport properties of systems consisting of self-propelled particles (SPP) have generated much attention lately [1–6]. This interest has been largely motivated by analogous processes taking place in numerous biological phenomena (e.g., bacterial migration on surfaces [7], flocking of birds, fish, quadrupeds [8], correlated motion of ants [9] and pedestrians [10]), as well as in various other systems, including driven granular materials [11,12] and traffic models [13]. The models describing these phenomena are distinctively nonequilibrium, exhibiting kinetic phase transitions and self-organization, and are of particular interest from the point of view of modern statistical mechanics [14].

In the simplest version of the SPP model [1]—introduced to study collective biological motion—each particle's velocity is set to a fixed magnitude, $v_0$. The interaction with the neighboring particles changes only the direction of motion: the particles tend to align their orientation to the local average velocity. Numerical simulations in 2D provided evidence of a second order phase transition [15] between an ordered phase in which the mean velocity of the entire system, $\langle v \rangle$, is nonzero and a disordered phase with $\langle v \rangle = 0$, as the strength of the noise is increased or the density of the particles decreased.

This SPP model is similar to the *XY* model of classical magnetic spins because the velocity of the particles, such as the local spin of the *XY* model, has fixed length and continuous rotational symmetry. In the $v_0 = 0$ and low noise limit the model reduces *exactly* to a Monte Carlo dynamics of the *XY* model. Since the *XY* model does *not* exhibit a long-range ordered phase at temperatures $T > 0$ [16], the ordered state observed in [1] is surprising. To explain this discrepancy, Toner and Tu (TT) [3] proposed a continuum theory that included in a self-consistent way the nonequilibrium effects as well. They have shown that their model is different from the *XY* model for $d < 4$ and found an ordered phase in $d = 2$ [17]. While TT provided the first theoretical demonstration of the ordered phase in 2D SPP models, the nonlinearity responsible for the long-range order in their continuum model is absent for $d = 1$.

Here we demonstrate that a kinetic phase transition and ordering takes place in 1D as well; i.e., the discrete $U \to -U$ symmetry breaks spontaneously. This result is as surprising as breaking of the rotational symmetry in 2D. This nonequilibrium phenomenon was not foreseen by the existing analytical approaches, which motivated us to introduce a new continuum theory describing the SPP model in one dimension. Numerical investigations indicate that the continuum theory and the discrete SPP model belong to the same universality class.

*The 1D SPP model.*—Let us consider $N$ off-lattice particles along a line of length $L$. The particles are characterized by their coordinate $x_i$ and dimensionless velocity $u_i$ updated as

$$x_i(t+1) = x_i(t) + v_0 u_i(t), \qquad u_i(t+1) = G(\langle u(t)\rangle_i) + \xi_i. \tag{1}$$

The local average velocity $\langle u \rangle_i$ for the $i$th particle is calculated over the particles located in the interval $[x_i - \Delta, x_i + \Delta]$, where we fix $\Delta = 1$ [18]. The antisymmetric function $G$ incorporates both the propulsion and friction forces which set the velocity in average to a prescribed value $v_0$: $G(u) > u$ for $0 < u < 1$ and $G(u) < u$ for $u > 1$ [19]. The distribution function $P$ of the noise $\xi_i$ is uniform in the interval $[-\eta/2, \eta/2]$.

Keeping $v_0$ constant ($v_0 = 0.1$), the adjustable control parameters of the model are the average density of the particles, $\rho = N/L$ and the noise amplitude $\eta$. We implemented one of the simplest choices [20] for $G$,

$$G(u) = \begin{cases} (u+1)/2 & \text{for } u > 0, \\ (u-1)/2 & \text{for } u < 0, \end{cases} \tag{2}$$

and applied random initial and periodic boundary conditions.

In Fig. 1 we show the time evolution of the model for $\eta = 2.0$. In a short time the system reaches an ordered state, characterized by a spontaneous broken symmetry and clustering of the particles. In contrast, for larger values of $\eta$ a disordered velocity field can be found.

*Scaling and exponents.*—To capture quantitatively the transition from an ordered to a disordered state, in Fig. 2a we plot the order parameter $\phi \equiv \langle u \rangle$ vs $\eta$ for various $\rho$. As in two dimensions [1,15], the ordered phase emerges through a second order phase transition. Near the critical

  

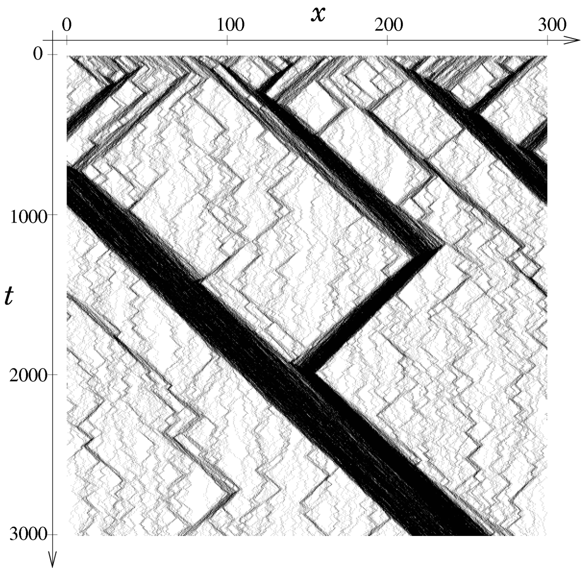


FIG. 1. The dynamics of the 1D SPP model for $L = 300$, $\eta = 2.0$, and $N = 600$. The darker gray level represents higher particle density. Note that the particles exhibit clustering and the spontaneous broken symmetry of motion.

noise amplitude, $\eta_c(\rho, L)$, which separates the ordered from the disordered phase, $\phi$ vanishes as

$$\phi(\eta,\rho) \sim \begin{cases} (\frac{\eta_c(\rho,L)-\eta}{\eta_c(\rho,L)})^{\beta} & \text{for } \eta < \eta_c(\rho,L)\,, \\ 0 & \text{for } \eta > \eta_c(\rho,L)\,, \end{cases} \quad (3)$$

whose finding is supported by the increasing scaling regime with the system size $L$ (Fig. 2b) and by the convergence of $\eta_c(\rho, L)$ to a nonzero $\eta_c(\rho, \infty)$ value for increasing system sizes. We find that $\beta = 0.60 \pm 0.05$, which is different from both the mean-field value $1/2$ [21] and $\beta = 0.42 \pm 0.03$ found in $d = 2$ [15].

Figure 2a also shows that the various $\phi(\eta, \rho)$ curves can be collapsed onto a single function $\phi_0(x)$, where $x = \eta/\eta_c(\rho)$, just like in $d = 2$. As shown in [15], the consequence of this fact is that near the critical density the order parameter vanishes as

$$\phi(\eta,\rho) \sim \begin{cases} (\frac{\rho-\rho_c(\eta,L)}{\rho_c(\eta,L)})^{\beta'} & \text{for } \rho > \rho_c(\eta,L)\,, \\ 0 & \text{for } \rho < \rho_c(\eta,L)\,, \end{cases} \quad (4)$$

with $\beta' = \beta$. These results can be summarized in the $\rho$-$\eta$ phase diagram shown in Fig. 2c. We also find that the critical line, $\eta_c(\rho)$, follows

$$\eta_c(\rho) \sim \rho^{\kappa}, \quad (5)$$

with $\kappa = 0.25 \pm 0.05$.

While the above numerical results demonstrate the existence of the phase transition in one dimension and provide numerical values for the scaling exponents $\beta, \beta'$, and $\kappa$, the origin of these values is unclear at this point. Ordering at finite noise level in our 1D model qualitatively can be interpreted by considering that due to the *biased motion* of the domains (groups of coherently moving particles), an effective long-range interaction is being built up during the development of the system. On the other hand, the emergence of the ordered phase in 1D is not predicted either by the equilibrium theories or by the TT model. This is why we introduce and investigate a set of continuum equations (which can be generalized to any dimension) in terms of $U(x,t)$ and $\varrho(x,t)$, where $U$ and $\varrho$ represent the coarse-grained dimensionless velocity and density fields, respectively. Analogous approaches were fruitful in the study of a similar SPP system, one-lane traffic flow [22].

*Continuum theory.*—Let us denote by $n(u,x,t)\,du dx$ the number of particles moving with a velocity in the range of $[v_0 u, v_0(u + du)]$ at time $t$ in the $[x, x + dx]$ interval. The particle density $\varrho(x,t)$ is then given as $\varrho = \int n\, du$, while the local dimensionless average velocity $U(x,t)$ can be calculated as $\varrho U = \int nu\, du$. According to the microscopic rules of the dynamics, in a given time interval $[t, t + \tau]$ all particles choose a certain velocity $v/v_0 = [G(\langle u\rangle) + \xi]$ and travel a distance $v\tau$. Thus, the time development of the ensemble average (denoted by the overline) of $n$ is governed by the master equation $\overline{n(u,x,t+\tau)} = \overline{\varrho(x',t)\, p(u\,|\,U(x',t))}$, where $x' = x - v_0 u\tau$ and $p(u\,|\,U)$ denotes the conditional probability of finding a particle with a velocity $u$ when the local velocity field $U$ is given. From Eq. (1) we have $p(u\,|\,U) = P(u - G(\langle U\rangle))$. Since $n$ is finite, the actual occupation numbers in a given system differ from $\overline{n}$. This fact can be accounted for by adding an intrinsic noise term to the master equation as

$$n(u,x,t+\tau) = \varrho(x',t)p(u\,|\,U(x',t)) + \nu(u,x',t)\,, \quad (6)$$

where $\nu$ has the following properties: (i) $\overline{\nu} = 0$; (ii) because of the conservation of the particles $\int \nu\, du = 0$; and (iii) since we have a random sampling process, the actual values of $n$ satisfy Poisson statistics, i.e., the distribution function of $\nu$ depends on $\varrho$, $u$, and $U$, as $P(\nu) = \lambda^{\nu+\lambda}\exp(-\lambda)/\Gamma(\nu + \lambda + 1)$, where $\lambda = \varrho\, p(u\,|\,U)$. Thus, we have $\overline{\nu^2} = \overline{n}$.

Taking the Taylor expansion of $n(u, x - v_0 u\tau, t)$ up to the second order in $x$ and integrating Eq. (6) according to $du$, in the $v_0\tau \ll 1$, $\sigma^2 \equiv \int P(u)u^2\, du \gg 1$, $\varrho \gg 1$, and $v_0\tau\sigma^2 < 1$ limit we obtain

$$\partial_t \varrho = -v_0 \partial_x(\varrho U) + D\partial_x^2 \varrho\,, \quad (7)$$

where $D = v_0^2\tau\sigma^2/2$. Note that the appearance of the diffusion term is a consequence of the nonvanishing correlation time $\tau$. Since $\int p(u\,|\,U)u\, du = G(\langle U\rangle)$, integrating Eq. (6) according to $udu$, expanding $\langle U\rangle$ as $\langle U\rangle = U + [\partial_x^2 U + 2(\partial_x U)(\partial_x \varrho)/\varrho]/6$ [23], using (7), we arrive at

$$\partial_t U = f(U) + \mu^2\partial_x^2 U + \alpha\,\frac{(\partial_x U)(\partial_x \varrho)}{\varrho} + \zeta\,, \quad (8)$$

where $f(U) = [G(U) - U]/\tau$, $\mu^2 = (dG/dU)/(6\tau)$, $\alpha = 2\mu^2$, and $\zeta = \int \nu u\, du/\varrho\tau$. Note that $f(U)$ is an antisymmetric function with $f(U) > 0$ for $0 < U < 1$ and $f(U) < 0$ for $U > 1$, $\overline{\zeta} = 0$, and $\overline{\zeta^2} = \sigma^2/\varrho\tau^2$.

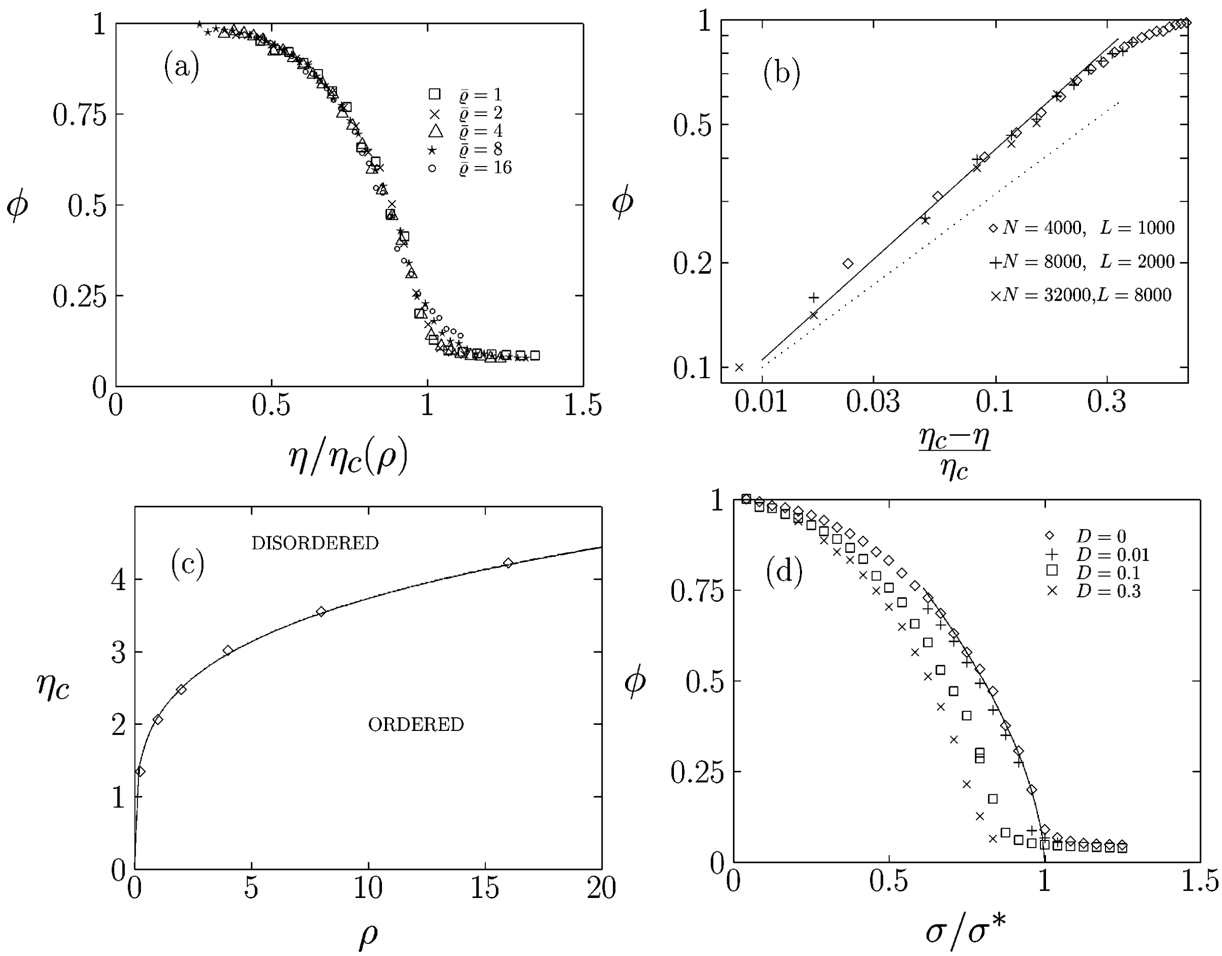

FIG. 2. (a) The order parameter $\phi$ vs the noise amplitude normalized by the critical amplitude $\eta_c(\rho)$, for $L = 1000$ and various values of $\rho$. For $\eta < n_c(\rho)$ the system is in a symmetry broken state indicated by $\phi > 0$. (b) $\phi$ vanishes as a power law in the vicinity of $\eta_c(\rho)$. Note the increasing scaling regime with increasing $L$. The solid line is a power-law fit with an exponent $\beta = 0.6$, while the dotted line shows the mean-field slope $\beta = 1/2$ as a comparison. (c) Phase diagram in the $\rho$-$\eta$ plane. The critical line follows $\eta_c(\rho) \sim \rho^\kappa$. The solid curve represents a fit with $\kappa = 1/4$. (d) The order parameter vs the standard deviation of the noise normalized by $\sigma^* = \sigma_c(D = 0)$, obtained by direct numerical integration of the continuum model for $\alpha = 2$, $\mu = 1$, $v_0 = 0.1$, $\rho = 1$, $L = 1000$, and various values of $D$. For $D \ll 1$ $\phi(\sigma)$ follows a power law with an exponent $\beta = 0.6$ (solid line).

At this point we consider Eqs. (7) and (8) with the coefficients $\mu$, $\alpha$, $\sigma$, $v_0$, and $D$ as the continuum theory describing a large class of SPP models. These equations differ from both the equilibrium field theories and the nonequilibrium system investigated by TT [3]. The main differences are due to (i) the nonlinear coupling term $(\partial_x U)(\partial_x \varrho)/\varrho$, and (ii) the statistical properties of the noise $\zeta$. For $\alpha = 0$ the dynamics of the velocity field $U$ is independent of $\varrho$, and Eq. (8) is equivalent to the time dependent Ginsburg-Landau $\Phi^4$ model describing spin chains, where domains of opposite magnetization develop at finite temperatures [21].

To study the effect of the nonlinear term in (8), we now investigate the development of the ordered phase in the deterministic case ($\sigma = 0$). For $\alpha = 0$ Eqs. (7) and (8) have a set of (meta)stable stationary solutions $\varrho^*$ and $U^*$ describing a "domain wall" separating two regions with opposite velocities. Since we can freely translate these solutions, we assume $U^*(0) = 0$. Performing linear stability analysis we next show that for certain finite values of $\alpha$ the above stationary solutions are unstable.

*Linear stability analysis.*—We make use of the fact that the dynamics of $\varrho$ is very slow compared to that of $U$, i.e., $v_0, D \ll 1$. We write $U$ in the form of $U(x,t) = U_0(x,t) + u(x,t)$, where $U_0(x,t) = U^*(x - \xi(t))$ and the position of the domain wall, $\xi(t)$, is defined by $U(\xi(t),t) = 0$. Now in the $u \ll U_0$, $\partial_x u \ll \partial_x U_0$, and $\xi \ll 1$ limit in the moving frame $x' = x - \xi(t)$, Eq. (8) reads as

$$\partial_t u' = \dot{\xi} a + (g - g_\infty) u' + \mu^2 \partial_x^2 u' + \alpha \xi h, \qquad (9)$$

where $u'(x') = u(x)$, $h = a \partial_x^2 \ln \varrho^*$, $a = \partial_x U^*$, and $g(x') = (df/dU)(U^*(x')) + g_\infty$ with $g_\infty = (df/dU)(1)$. From $U(\xi(t),t) = U_0(\xi(t),t) = 0$ we get $u'(x' = 0, t) = 0$, which yields

$$-\dot{\xi} a(0) = \mu^2 \partial_x^2 u'(0,t) + \alpha \xi h. \qquad (10)$$

Now (9) and (10) describe the time development of the velocity field in terms of $u'$ and $\xi$.

Fourier transforming (9) and (10) we find that the short wavelength fluctuations ($k \to \infty$) are stabilized by the Laplacian term in (9). However, the growth rate $\lambda$ of the long wavelength fluctuations, defined by $\xi \sim \exp(\lambda t)$, is positive for $\alpha > \alpha_c \geq 0$, where $\alpha_c$ depends on $D$, $v_0$ and the functional form of $f$. Thus for $\alpha > \alpha_c$ the domain wall solution $U^*$ is unstable: it moves in a direction selected by the initial asymmetry of the perturbation. Hence at $t \to \infty$ the walls disappear and all particles move in the same direction. It is also easy to see that the $U = \pm 1$ solutions are absolute stable against small perturbations; thus it is plausible to assume that the system converges into those solutions even for finite noise. In the case of a phase transition one expects that the walls become unstable. The result of the above analysis is consistent with such an expectation; however, this kind of linear stability analysis is not able to demonstrate directly the existence of a continuous phase transition in the presence of fluctuations.

To further confirm the relevance of the continuum theory to the discrete model (1), we integrate numerically (7) and (8). The results of the integration are in excellent agreement with the results obtained for the discrete model and with the linear stability analysis and can be summarized as follows: (i) For the noiseless case ($\sigma = 0$) we find that for $\alpha > \alpha_c$ there is an ordered phase, which disappears for $\alpha < \alpha_c$; (ii) the ordered phase is present for the noisy $\sigma > 0$ as well; (iii) increasing $\sigma$ leads to a second order phase transition from the ordered to the disordered state. Since $\sigma$ plays the role of $\eta$ in (1), this transition is equivalent with (3) observed in the discrete model; (iv) finally, we measured the order parameter $\phi$ as a function of $\sigma$ for various values of $D$. As Fig. 2d illustrates, for small $D$, Eq. (3) with $\beta = 0.6$ provides an excellent fit to the numerical results, indicating that the discrete model (1) and the continuum theory (7) and (8) *belong to the same universality class.*

In conclusion, we showed that the SSP model exhibits spontaneous symmetry breaking and ordering in one dimension, a result surprising in the light of the equilibrium spin models. We introduced a new continuum theory, whose terms are explicitly derived from the ingredients of the discrete model (in contrast with constructing it from symmetry arguments). Linear stability analysis indicates that an ordered phase can develop as the domain walls become unstable.

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