Global versus Local Billiard Level Dynamics: The Limits of Universality

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(Received 17 November 1998)

Level dynamics measurements have been performed in a Sinai microwave billiard as a function of a single length, as well as in rectangular billiards with randomly distributed disks as a function of the position of one disk. In the first case the field distribution is changed *globally*, and velocity distributions and autocorrelation functions are well described by universal functions derived by Simons and Altshuler. In the second case the field distribution is changed *locally*. Here another type of universal correlation is observed. It can be derived under the assumption that chaotic wave functions may be described by a random superposition of plane waves. [S0031-9007(99)08639-1]

PACS numbers: 05.45.–a, 73.23.–b

In 1972 Edwards and Thouless noticed that the conductivity of a disordered system is closely related to the sensitivity of its eigenvalues on an external perturbation [1,2]. For a ring with a perpendicularly applied magnetic field they conjectured that the conductivity C is proportional to the averaged curvature of the eigenvalues, $C \sim \langle |\frac{\partial^2 E_n}{\partial \varphi^2}|_{\varphi=0}\rangle$, where φ is the magnetic flux through the ring. In 1992 Akkermans and Montambaux showed that the conductivity may alternatively be expressed in terms of the eigenvalue velocities, $C \sim \langle |\frac{\partial E_n}{\partial \varphi}|^2\rangle$ [3]. This suggests to rescale the parameter and the eigenvalues by

$$x = \frac{1}{\Delta}\left\langle\left|\frac{\partial E_n}{\partial \varphi}\right|^2\right\rangle^{1/2}\varphi, \qquad \epsilon_n(x) = \frac{E_n(\varphi)}{\Delta}, \tag{1}$$

where Δ is the mean level spacing. Szafer, Simons, and Altshuler studied a number of parametric correlations of the rescaled eigenenergies [4,5], in particular the velocity autocorrelation function

$$c(x) = \left\langle \frac{\partial \epsilon_n(X+x)}{\partial X}\frac{\partial \epsilon_n(X)}{\partial X}\right\rangle - \left\langle\frac{\partial \epsilon_n(X)}{\partial X}\right\rangle^2, \tag{2}$$

originally introduced by Yang and Burgdörfer [6], and conjectured a universal behavior as long as the so-called zero-mode approximation holds, i.e., in the range where the energy fluctuations show random matrix behavior. For the velocity distribution Simons and Altshuler found a Gaussian behavior [5]. The same behavior has been obtained by a completely different approach starting from the analogy between the level dynamics of a chaotic system and the dynamics of a one-dimensional gas with repulsive interaction [7,8]. In the region of onset of localization deviations from the Gaussian behavior are found [9].

Since in the zero-mode approximation the energy correlations of a disordered system are identical to those of random matrices, it came as no surprise that the universal behavior of parametric correlations was found in billiard systems as well [5]. Universal behavior was observed also for the hydrogen atom in a strong magnetic field [10], conformally deformed [11] and ray-splitting billiards [12], and in the acoustic spectra of vibrating quartz blocks [13]. In all cases the general features of the conjectured universal behavior had been reproduced reasonably well, but a number of significant discrepancies remained unexplained.

This was our motivation to study different types of billiard level dynamics in a bit more detail. All results to be presented below have been obtained in microwave billiards [14]. Here it is sufficient to note that for flat resonators the electromagnetic spectrum is completely equivalent to the quantum mechanical spectrum of the corresponding billiard, as long as one does not surpass the frequency $\nu_{\max} = c/2h$, where h is the resonator height. In the experiments we choose $h = 8$ mm yielding a maximum frequency of 18.74 GHz.

One of the systems studied was a quarter Sinai billiard with a width $b = 200$ mm, a radius $r = 70$ mm of the quarter circle, and a length a which was varied between 480 and 500 mm in steps of 0.2 mm. About 120 eigenvalues entered into the data analysis in the frequency range 14.5 to 15.5 GHz. The second system was a rectangular billiard with side lengths $a = 340$ mm, $b = 240$ mm, containing 20 randomly distributed circular disks with a diameter of 5 mm (see Fig. 1). By a

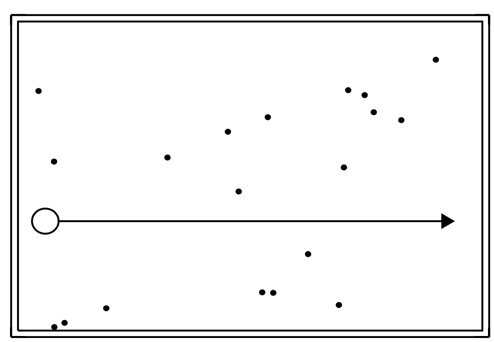

FIG. 1. Sketch of the billiard used for the local level dynamics (to scale).

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spatially resolved measurement [15] we found that all eigenfunctions ψ in the studied frequency range were delocalized and $|\psi|^2$ was Porter-Thomas distributed. The position of one of the disks was varied in one direction in steps of 1 mm. Whereas the first type of level dynamics may be considered as global, since a shift of the billiard length of the order of 1 wavelength will change the wave-function pattern everywhere in the billiard, the shift of the disk gives rise to a local modification only.

We start with a discussion of the global level dynamics. Figure 2 shows the velocity distribution for the quarter Sinai billiard with length a as the level dynamics parameter. The distribution is well described by a Gaussian in accordance with the expected universal behavior (this result has been presented already in [8]). Figure 3 shows the corresponding velocity correlator. To obtain the result, each eigenvalue was studied over a range of four to five avoided crossings, and the scaling was performed by calculating the mean squared velocity for each eigenvalue independently. Subsequently, the results of about 120 eigenvalues were superimposed. The solid line corresponds to Simons' and Altshuler's universal function [16]. The overall agreement between experiment and theory is good, but for $x > 2.5$ (not shown) the correlation function does not approach zero but stays at negative values. This is an artifact resulting from an insufficient number of data points making the calculation of the average $\langle \frac{\partial\epsilon_n(X+x)}{\partial X} \frac{\partial\epsilon_n(X)}{\partial X} \rangle$ unreliable for large x values. Most correlation functions found in the literature end at x values of at most 1.5, probably just for this reason.

Let us now turn to the discussion of the local level dynamics, where the position of one disk was varied. Whether a level dynamics must be considered as global or local, depends on the parameter $\delta = kD$, where D is the diameter of the disk, and k the wave number. It is well known that in the limit of small δ values the spectral properties of billiards containing hard spheres deviate significantly from random matrix behavior [17]. Figure 4 shows the velocity distributions for three different δ ranges. In Fig. 4(a) a disk with $D = 5$ mm was used, and the eigenvalues were taken in the frequency range 3.4 to 6 GHz. In Figs. 4(b) and 4(c) the diameter of the movable disk was $D = 20$ mm with eigenvalues in the frequency ranges 3.4 to 6 GHz and 12.5 to 14.5 GHz, respectively. None of the found velocity distributions is Gaussian. One observes instead a distribution with a pronounced peak at $v = 0$, decreasing only exponentially for large values of $|v|$. With increasing δ values the distributions turn gradually into a Gaussian. We completed the series by a level dynamics measurement for a half Sinai billiard, where the position of the half circle was varied. Here the obtained velocity distribution (not shown), corresponding to δ values between 30 and 37, was already close to a Gaussian distribution.

Figure 5 shows the corresponding velocity autocorrelation functions. The scaling technique applied was the same as above. There is no longer any similarity between the experimental curves and the universal function. Only for the largest δ value displayed, the experimental curve seems to approach the Simons-Altshuler correlation function again.

The results can be understood, if the movable disk is interpreted as a perturber probing the field in the resonator (the perturbing bead method has been used many years ago to map the field distributions in microwave cavities [18], and has recently been applied to the study of wave functions in chaotic billiards as well [19–21]). In two-dimensional billiards the insertion of a metallic perturber leads to a negative frequency shift proportional to E^2, where E is the electric field strength in the resonator in the absence of the perturber. This holds as long as the dimensions of the perturber are small compared to the wavelength, i.e., in the limit $\delta \rightarrow 0$. Applied to the present problem this means that the eigenvalue velocity

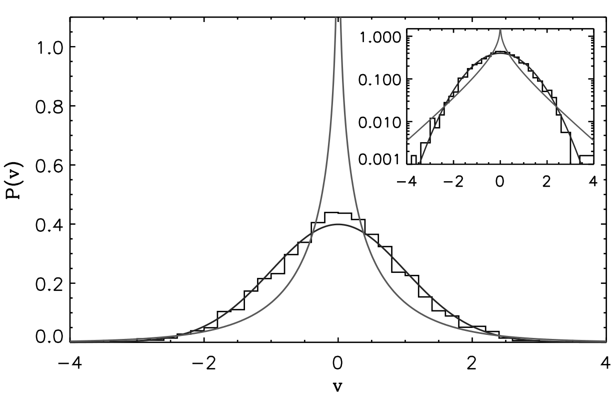

FIG. 2. Velocity distribution in a quarter Sinai billiard with one length as the level dynamics parameter. The solid lines correspond to a Gaussian distribution and a distribution described by a modified Bessel function [see Eq. (5)], respectively. The inset shows the distribution in a logarithmic scale.

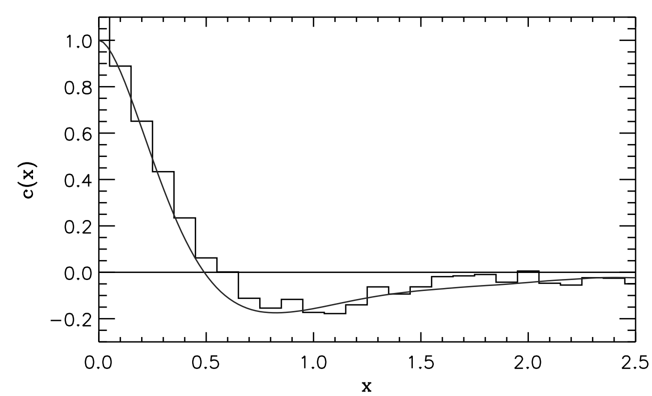

FIG. 3. Velocity autocorrelation function in a quarter Sinai billiard, where the level dynamics parameter was scaled according to Eq. (1). No less than 2000 velocity pairs entered every bin of the histogram. The solid line corresponds to the universal autocorrelation function of Simons and Altshuler.

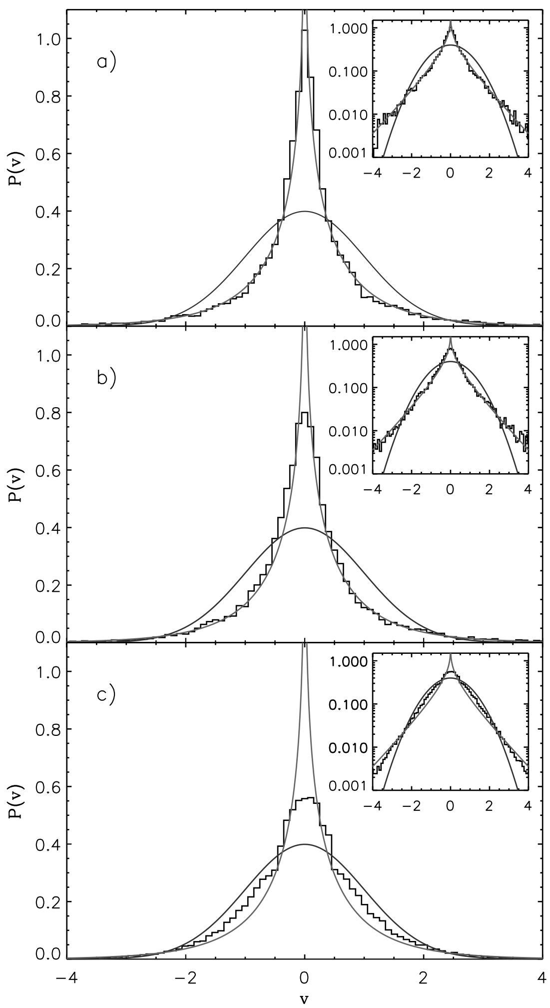

FIG. 4. Velocity distributions in a rectangular billiard with randomly distributed disks with the position of one disk with diameter D as the level dynamics parameter. The ranges of $\delta = kD$ are $0.35 < \delta < 0.65$ (a), $1.4 < \delta < 2.6$ (b), and $5.1 < \delta < 5.9$ (c).

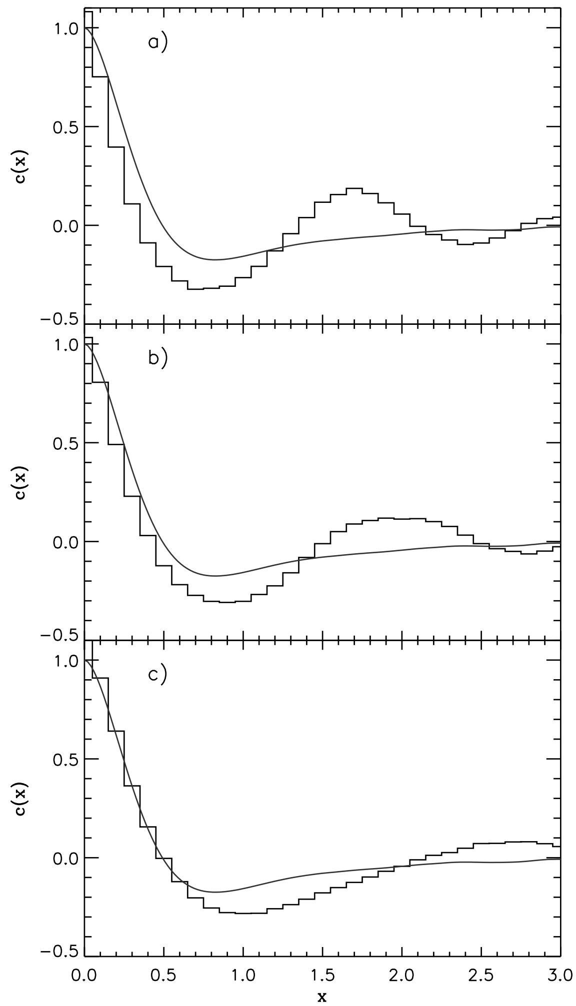

FIG. 5. Velocity autocorrelation functions in a rectangular billiard with randomly distributed disks, where the level dynamics parameter was scaled according to Eq. (1). The ranges of δ are the same as in Fig. 4.

is given by $\partial E_n/\partial r = \alpha \nabla |\psi|^2$ where ∇ is the gradient in the direction of the displacement, and α is a constant depending on the geometry of the perturber. It follows for the velocity distribution function

$$P(v) = \langle \delta(v - 2\alpha\psi\nabla\psi)\rangle\,. \tag{3}$$

Under the assumption that the wave functions can be described by a random superposition of plane waves [22], ψ and $\nabla\psi$ are uncorrelated, and Gaussian distributed [23],

$$P_1(\psi) = \sqrt{\frac{A}{2\pi}}\, e^{-(A\psi^2/2)},$$
$$P_2(\nabla\psi) = \sqrt{\frac{A}{2\pi k^2}}\, e^{-(A(\nabla\psi)^2/2k^2)}. \tag{4}$$

The influence of the boundary is negligible here, since the linear dimensions of the billiard exceed the typical wavelength by factors of 5 to 10. Using Eq. (4) the average (3) is easily calculated and yields

$$P(v) = \frac{\beta}{\pi}\, K_0(\beta|v|)\,, \tag{5}$$

where $K_0(x)$ is a modified Bessel function, and $\beta = A/2\alpha k$. The solid lines plotted in addition to the Gaussian curves in Figs. 2 and 4 have been calculated from Eq. (5). In the limit of small δ values distribution (5) describes the experimental distributions perfectly.

The influence of local perturbations on the energy levels has been studied by Aleiner and Matveev [24], who derived an explicit expression for the joint distribution function of initial and final energy levels. In their model the velocities are Porter-Thomas distributed [25], if the coupling strength is taken as the level dynamics parameter. The same distribution would have been expected in our case, if the coupling strength α would have been

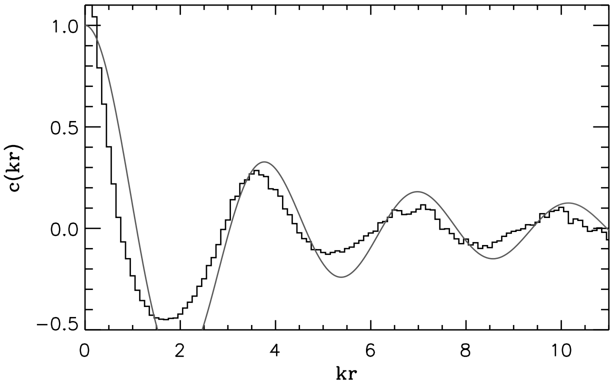

FIG. 6. Same as Fig. 5(a), but with the level dynamics parameter scaled according to Eq. (8).

varied instead of the position (which, however, would be technically difficult to realize).

For the quadratical average of the eigenvalue velocities we obtain using Eq. (5)

$$\left\langle \left(\frac{\partial E_n}{\partial r} \right)^2 \right\rangle = \frac{1}{\beta^2} \,. \tag{6}$$

Entering with this expression into Eq. (1), we get for the rescaled parameter

$$x = \frac{1}{\Delta \beta} r = \frac{\alpha}{2\pi} kr \,, \tag{7}$$

where we have used that in billiards the mean level spacing is given by $\Delta = 4\pi/A$. Equation (7) shows that for the local level dynamics x is not an universal parameter, since it depends via α on the geometry of the movable disk. We shall therefore use the rescaled parameter

$$\bar{x} = kr \tag{8}$$

instead in the following. From the approach of random superposition of plane waves [22] the velocity autocorrelation function can be easily calculated, too. Using standard techniques as they are described, e.g., in Ref. [26], we get

$$c(\bar{x}) = -[J_0^2(\bar{x})]'' = J_0^2(\bar{x}) - 2J_1^2(\bar{x}) - J_0(\bar{x})J_2(\bar{x}) \,. \tag{9}$$

Figure 6 shows again the velocity autocorrelation of Fig. 5(a) for the local level dynamics, but now as a function of $\bar{x}$. The solid line corresponds to the theoretical expectation (9). The experimental curve follows closely the predicted oscillations. With increasing δ the oscillations are more and more damped, but the wavelength is still in accordance with the theory (not shown).

This paper has shown that two different regimes of level dynamics have to be discriminated. In the local regime velocity distributions and autocorrelation functions are quantitatively described by the approach of random superposition of plane wave, if the scaling (8) is applied. In the global regime, on the other hand, Simons' and Altshuler's universal functions describe the experimental results well, and the scaling (1) is the appropriate one. The parameter $\delta = kD$ governs the transition between the two regimes.

Discussions with Y. V. Fyodorov and T. Guhr at different stages of the experiments are gratefully acknowledged. We thank H. U. Baranger for calling our attention to Ref. [23], and E. R. Mucciolo for making his calculations of the universal velocity correlator available to us. The experiments were supported by the Deutsche Forschungsgemeinschaft via the SFB 185 "Nichtlineare Dynamik" and by an individual grant.

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