Chaos and Synchronized Chaos in an Earthquake Model

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We show that chaos is present in the symmetric two-block Burridge-Knopoff model for earthquakes. This is in contrast with previous numerical studies, but in agreement with experimental results. In this system, we have found a rich dynamical behavior with an unusual route to chaos. In the three-block system, we see the appearance of synchronized chaos, showing that this concept can have potential applications in the field of seismology. [S0031-9007(98)08083-1]

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In recent years, the phenomenon of chaotic synchronization has been a subject of intensive studies. By definition, chaotic systems present strong sensitivity to the initial conditions, and in principle it seems impossible to synchronize them. However, Fujisaka *et al.* [1] and Pecora *et al.* [2] showed that systems with chaotic behavior can be synchronized, if appropriate connections among them are made. This phenomenon has been called "chaotic synchronization," and has been investigated mainly in applications for secure communications [3].

Another area of active research nowadays is related to systems that present avalanchelike dynamics. This was triggered by a paper by Bak, Tang, and Wiesenfeld [4]. They showed that certain dissipative systems, with many degrees of freedom, naturally evolve to a critical state characterized by power-law distributions in space and time. They denoted this phenomenon self-organized criticality (SOC).

One of the systems that has been studied in connection with SOC is the Burridge-Knopoff (BK) model for earthquakes [5]. This model consists of blocks connected by springs. The whole system is pulled with constant velocity on a surface with friction. It has been shown experimentally [5] and numerically [6] that this model presents a region of power-law distribution similar to what is observed in real earthquakes, namely, the Gutenberg-Richter law [7]. Since the power law does not span the entire system, one could say that this system does not present what has been defined as SOC. However, a variation of it, called the "train model," does present SOC [8,9].

After the work by Carlson and Langer [6], several studies on the BK model were performed. With respect to the chaotic properties of the model, we are aware of the numerical studies by Nussbaum and Ruina [10], Huang and Turcotte [11], Nakanishi [12], Crisanti *et al.* [13] and Lacorata *et al.* [14]. In [12] and [13] the systems considered had more than two blocks and they were evolved by cellular automaton rules. Nussbaum *et al.* studied a symmetric two-block BK model, and verified that, with a friction force of the Coulomb type (that is the dynamic friction coefficient being constant), the system presents only period behavior. Huang and Turcotte, and Lacorata and Paladin found chaotic behavior in the two-block BK model only with the presence of an asymmetry in the system, even considering a velocity weakening friction force [11,14]. In particular, they considered the friction force in one block being different from the friction force in the other block. On the other hand, by modeling the two-block BK model by electronic circuits, Field, Venturi, and Nori [15] showed experimentally that a completely *symmetric* system does present chaotic behavior in a wide range of the parameter space. Therefore, their results are in contradiction to what was reported in [11,14]. One of the purposes of this Letter is to resolve this contradiction. We show that the two-block BK system in a symmetric configuration is chaotic. As in the experimental study, chaos is seen in a wide range of parameter values. We stress that a one-block system in the BK model cannot present chaos, since its dimensionality is smaller than the minimum dimension necessary for a system to present chaotic behavior, which is three. (It is obvious that here we are considering the absence of elements with delay in the system, since in this way one could increase its dimensionality up to infinity.)

In this Letter we also show that the phenomenon of synchronized chaos appears in the three-block system of the Burridge-Knopoff model for earthquakes. Most importantly, it comes naturally from the geometry of the system, without any need for special connections, as is generally the case in the studies of synchronized chaos. From our results, we speculate that synchronized chaos may have applications in the field of seismology. That is, earthquakes faults, which are generally coupled through the elastic media in the Earth crust could in principle synchronize, even when they have an irregular (chaotic) dynamics. As a consequence of synchronization, the dimensionality of the attractor of the system decreases what simplifies the analysis of the system, as discussed below. We quantify the degree of synchronization by studying the Liapunov exponent associated with the synchronization manifold, that is, the transverse (or conditional) Liapunov exponent [1,16], and compare it with the Liapunov exponents of the three-block system.

We start by reviewing the BK model. It consists of a chain of blocks of mass m connected by coil springs of strength k_c to its nearest neighbors. They are situated on a rough surface. Between the blocks and the surface there is a frictional force F. Here we consider that F is a function of the block velocity. Each block is also attached by a leaf spring of strength k_p to a surface that moves with constant velocity v. A figure of the system can be found in [6]. Following Carlson and Langer [6], we denote by X_j the position of block j with respect to its equilibrium position, write the friction force as $F(\dot{X}_j/v_c) = F_0\Phi(\dot{X}_j/v_c)$, where $\Phi(0) = 1$ and v_c is a characteristic velocity, and introduce the variables $\tau \equiv \omega_p t$, $\omega_p^2 \equiv k_p/m$, $U_j \equiv k_p X_j/F_0$. Then the equation of motion for the two-block system can be written in the following dimensionless form:

$$\begin{aligned} \ddot{U}_1 &= k(U_2 - U_1) - U_1 + \nu\tau - \Phi(\dot{U}_1/\nu_1^c), \\ \ddot{U}_2 &= k(U_1 - U_2) - U_2 + \nu\tau - \Phi(\dot{U}_2/\nu_2^c), \end{aligned} \tag{1}$$

with $\nu \equiv v/V_0$, $\nu^c \equiv v_c/V_0$, $V_0 \equiv F_0/\sqrt{k_p m}$, and $k \equiv k_c/k_p$. Dots denote differentiation with respect to τ. Equation (1) is valid only when block j is moving. If it is not, its equation is simply $\dot{U}_j = \nu$. We use the velocity weakening friction force introduced in [6], given by

$$\Phi(\dot{U}/\nu^c) = \frac{1}{1 + \dot{U}/\nu^c}, \tag{2}$$

which is a simple nonlinear function. In the simulations displayed here, we did not allow backward motion, that is, the static friction force can take any value to prevent it. However, several tests showed that if backward motion is allowed, the results remain essentially the same.

In contradiction with the reports in [11,14], but in agreement with experimental results [15], we find that a velocity weakening friction force leads to a rich dynamical behavior in the two-block BK system, even when it is identical for the two blocks. We find the presence of periodic, quasiperiodic, and chaotic behavior. In Fig. 1(a) we show an example of chaotic motion, by plotting $\dot{U}_1$ versus $U_1 - U_1^e$, with $1/\nu^c \equiv 1/\nu_1^c = 1/\nu_2^c = 1$. We denote $U_j^e \equiv \nu\tau - \nu^c/(\nu^c + \nu)$ the unstable equilibrium point around which the orbits of block j circle in phase space, which is found by taking $\ddot{U}_j = 0$ and $\dot{U}_j = \nu$ in Eq. (1) (the stability of such a solution for any number of blocks was analyzed in [6]). Unless explicitly stated otherwise, we take in the numerical studies shown here as $k = 1$ and $\nu = 0.1$. However, similar behavior was found for other parameter values as well. We do not display $\dot{U}_2$ versus $U_2 - U_2^e$, since we have found that for this system the attractors of the two blocks are the same. In other words, the plot of $\dot{U}_2$ versus $U_2 - U_2^e$ is identical to the one of $\dot{U}_1$ versus $U_1 - U_1^e$, showing that the two blocks will visit the same region of the phase space, but not necessarily at the same time. The initial conditions for the simulations shown here are the blocks initially at rest and with small random displacements from their equilibrium position. In all the simulations, a transient time was discarded.

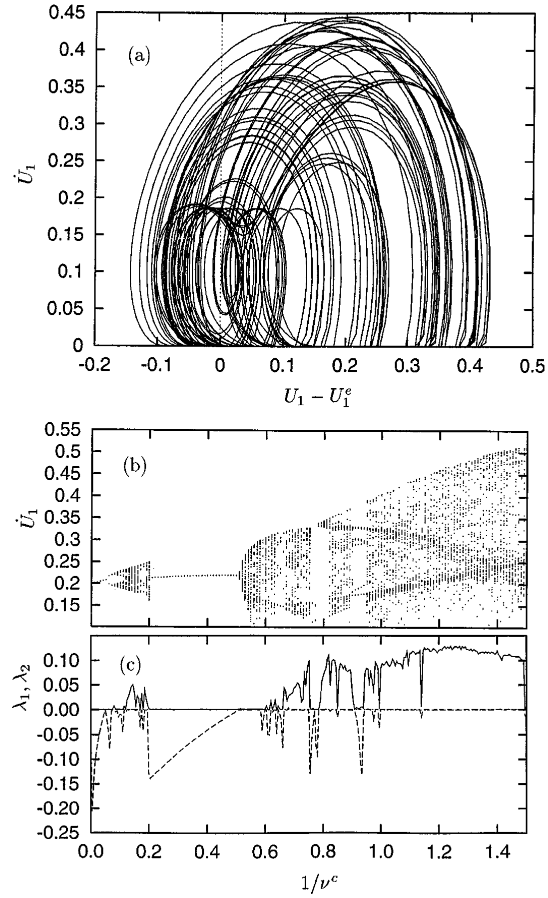

FIG. 1. (a) Orbit in phase space for block 1 when $1/\nu_1^c = 1/\nu_2^c = 1$. (b) Bifurcation diagram of $\dot{U}_1$ in the Poincaré section in which $U_1 - U_1^e = 0$. (c) The largest Liapunov exponent λ_1 (solid line) and the second-largest Liapunov exponent λ_2 (dashed line) of the system. Here we have $\nu = 0.1$ and $k = 1$ in a two-block system. The Liapunov exponents were calculated with perturbations of 10^{-4}, time step of 0.05, and 400 000 iterations.

The bifurcation diagram for the two-block system is shown in Fig. 1(b). There we display $\dot{U}_1$ versus $1/\nu^c$, in the Poincaré section satisfying $U_1 - U_1^e = 0$. In order to quantitatively characterize the dynamics, we have calculated the two largest Liapunov exponents of the system. If the largest Liapunov exponent (LLE) of the system is greater than zero, then, by definition, the system is chaotic. Quasiperiodic motion occurs in this system when the LLE and the second-largest Liapunov exponent (SLLE) are zero, and the motion is periodic when the LLE is zero and the SLLE is negative. We used the method introduced in [17] to calculate the LLE and SLLE, and our results are displayed in Fig. 1(c). The solid line refers to the LLE and the dashed line is the SLLE. We investigated in detail the first entrance into chaos, which occurs at $1/\nu^c \approx 0.112$ for these parameter values and noticed that it is unusual. We found that the route to chaos is from period one to period two and then directly into chaos. This unusual route is probably due to the fact that here we have a system governed by nondifferentiable flows. Most of what is known in dynamical system theory deals with systems that are infinitely differentiable, which is not the case here. More details about return maps and other quantities for this system will be published elsewhere [18].

One may ask why Huang and Turcotte [11] did not find chaotic behavior in the symmetric BK model? The reason is that they made an inconsistent assumption, that is, that the driving block does not move during the slipping events. This assumption is equivalent to taking the pulling velocity going to zero, since they drop out the term $\nu\tau$ in Eq. (1). However, from the equations of motion of the system, it turns out that the blocks will not move, if they are initially at rest, if one considers this, since the displacement of the blocks is proportional to the pulling velocity. This is shown in detail in [6]. One way to avoid this is to take a discontinuous friction force as in [19], and this was not considered in the study of the symmetric system by Huang and Turcotte. The system equation in [14] is, in our viewpoint, technically correct. The reason why they did not see chaos in the symmetric system is probably because they did not study a large enough region of the parameter space.

Now we concentrate on the phenomenon of chaotic synchronization in the BK model. We have not found this phenomenon in the two-block system with the friction force we consider here. If the two blocks were synchronized, they would behave as a single block, and this would be equivalent to having a single block system with chaotic behavior, which we know is impossible, as discussed above. We see, however, that a three-block system does present chaotic synchronization.

The equation of motion for the three-block system in dimensionless form is

$$\begin{aligned} \ddot{U}_1 &= k(U_2 - U_1) - U_1 + \nu\tau - \Phi(\dot{U}_1/\nu_1^c)\,, \\ \ddot{U}_2 &= k(U_1 - 2U_2 + U_3) \\ &\quad - U_2 + \nu\tau - \Phi(\dot{U}_2/\nu_2^c)\,, \\ \ddot{U}_3 &= k(U_2 - U_3) - U_3 + \nu\tau - \Phi(\dot{U}_3/\nu_3^c)\,. \end{aligned} \tag{3}$$

We see the equations that govern the motion of blocks 1 and 3 have the same functional form. They are also linked to a common subsystem, that is, block 2. This configuration is not of the "master-slave" type, since there is feedback between blocks 1 and 2 and between blocks 2 and 3. However, the equations have the necessary ingredients for chaotic synchronization between blocks 1 and 3 to occur, which is the same functional form.

We have found chaotic synchronization when the parameters for all the blocks are the same only in a very small region of the parameter space. It is not difficult to understand why. With the leaf springs having the same value, blocks 1 and 3 are more loose than block 2 (since the latter is attached to two coil springs instead of one). Therefore, in general, blocks 1 and 3 attain larger velocities than block 2, and with the friction force that we use, they are more unstable. Chaotic synchronization happens when the subsystems to be synchronized are more stable than the subsystem to which they are connected [2]. Also, it is a property of chaotic synchronization that blocks 1 and 3 need to have the same parameter values, if perfect synchronization in the absence of control is the goal.

However, we find that when there is an asymmetry in the system, chaotic synchronization occurs in a large range of the parameter space. We can either introduce asymmetries in the springs, masses, or friction forces. We choose the latter to demonstrate our results, but similar results were found in the other two cases. Therefore, we make the friction force in block 2 smaller than the friction force in blocks 1 and 3 (which means that they are more rough than block 2).

In Fig. 2(a) we show an example of a chaotic orbit for block 1, with the parameter values $1/\nu^c \equiv 1/\nu_1^c = 1/\nu_3^c = 4/\nu_2^c = 0.165$. We will see that this attractor is from a regime in which blocks 1 and 3 are synchronized. In Fig. 2(b) the LLE (solid line) and the SLLE (dashed line) of the three-block system are shown. In Fig. 2(c) we show the Liapunov exponent of the synchronization manifold (solid line), that is, what is called the transverse (or conditional) Liapunov exponent [1,16]. We have calculated the transverse Liapunov exponent by adapting the method introduced by Benettin [17]. That is, after the transient dies out, we evolve the orbit of block 3 by making it slightly different from the orbit of block 1. Then we verify how the difference between the orbits of the two blocks evolves after a short time step. The perturbation is renormalized in the direction of the maximum growth, and the process is repeated many times. The transverse Liapunov exponent is given by the average logarithm (in this paper we use base 2) of the growth (or shrinkage) of the perturbation along the orbit. Figure 2(c) also shows (dashed line) the Euclidean distance D in phase space between blocks 1 and 3, that is, $D \equiv \sqrt{(U_1 - U_3)^2 + (\dot{U}_1 - \dot{U}_3)^2}$, as a function of $1/\nu^c$. This distance is an average over a time $\Delta\tau = 2000$. We see that the transverse Liapunov exponent correctly determines the region in which blocks 1 and 3 are synchronized. In this situation, the transverse Liapunov exponent is negative and the distance between the two blocks is zero. Comparing Figs. 2(b) and 2(c) we can identify the regions of chaotic synchronization, where we have one or more Liapunov exponents of the three-block system greater than zero and the transverse Liapunov exponent is less than zero. This is, for example, the case of the attractor shown in Fig. 2(a). There only one Liapunov exponent of the system is larger than zero. The comparison between Figs. 2(b) and 2(c) also shows that the transition from chaos to hyperchaos [20] does not determine here the transition from chaotic synchronization to nonsynchronization, as was the case of the system studied in [21] (the hyperchaos regime is defined as the one in which there is more than one positive Liapunov exponent). For example, at $1/\nu^c = 0.38$ there is only one positive Liapunov exponent, and no synchronization is seen between blocks 1 and 3. There is also the case in which the LLE and the SLLE

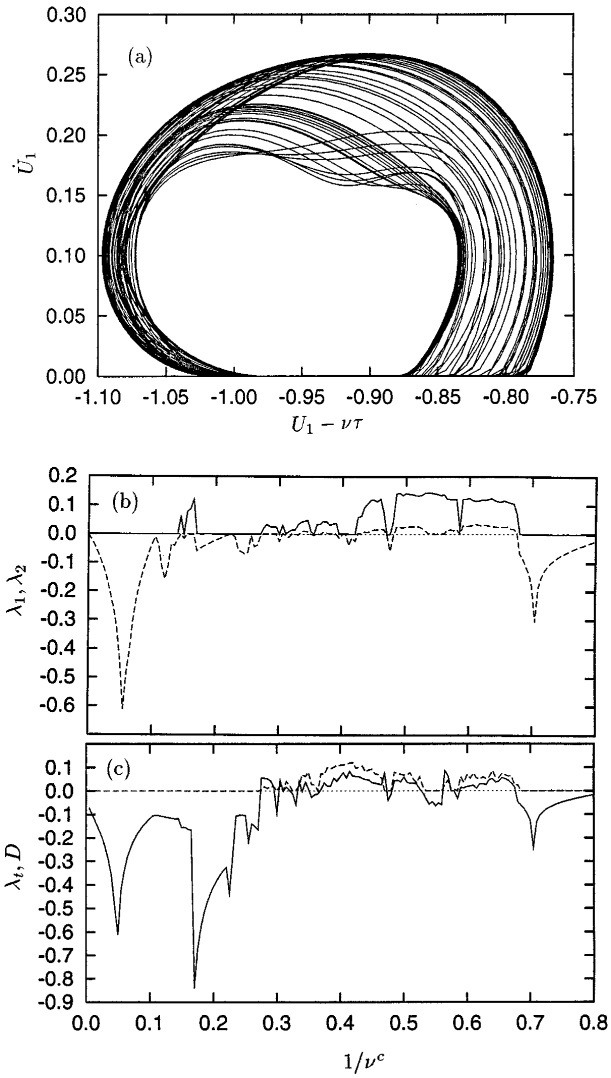

FIG. 2. (a) Orbits in phase space for $\dot{U}_1$ versus $U_1 - \nu\tau$ when $1/\nu^c \equiv 1/\nu_1^c = 1/\nu_3^c = 4/\nu_2^c = 0.165$. (b) The largest Liapunov exponent λ_1 (solid line) and the second-largest Liapunov exponent λ_2 (dashed line) as a function of $1/\nu^c$. (c) The transverse Liapunov exponent λ_t (solid line) and the Euclidean distance D in phase space between blocks 1 and 3 (dashed line) as functions of $1/\nu^c$. Here we have $\nu = 0.1$, $k = 1$ in a three-block system. The Liapunov exponents were calculated with perturbations of 10^{-4}, time step of 0.05, and 400 000 iterations. The dotted line at $D = 0$ is a guide to the eye.

are nonpositive, and blocks 1 and 3 do not synchronize, as in $1/\nu^c = 0.4$.

Now we discuss what is the possible relevance of our finds to the analysis of real earthquakes. It is known that earthquakes are not strictly periodic phenomena. If their irregular behavior is caused by a chaotic, deterministic process, as this simple spring-block model suggests, then, in principle, prediction about them could be made in a short time scale. In this way, the damage they cause could be minimized. The problem with modeling chaotic systems is that, if the dimension of their attractor is very large, very little can be done with respect to prediction [22]. Such systems are treated basically as stochastic systems. However, if synchronization happens among elements of a large system, then the dimensionality of the attractor decreases. For example, in the three-block system studied here, the dimension of the attractor can decrease by up to 2. If the dimension of the attractor decreases, the analysis of the system becomes easier. This fact emphasizes the need for a deeper analysis of the relationship between synchronized chaos and prediction. We hope that this work will motivate more studies on chaotic synchronization for applications to seismology.

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