Phase Mixing of Nonlinear Plasma Oscillations in an Arbitrary Mass Ratio Cold Plasma

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Nonlinear plasma oscillations in an arbitrary mass ratio cold plasma have been studied using 1D particle-in-cell simulation. In contrast to earlier work for infinitely massive ion plasmas it has been found that the oscillations phase mix away at any amplitude and that the rate at which phase mixing occurs depends on the mass ratio ($\Delta = m_-/m_+$) and the amplitude. A perturbation theoretic calculation carried up to third order predicts that the normalized phase mixing time $\omega_{p-} t_{\text{mix}}$ depends on the amplitude A and the mass ratio Δ as $\sim [(A^2/24)(\Delta/\sqrt{1+\Delta})]^{-1/3}$. We have confirmed this scaling in our simulations. These cold plasma results may have direct relevance to recent experiments on superintense laser beam plasma interactions with applications to particle acceleration, fast ignitor concept, etc. [S0031-9007(99)08544-0]

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The physics of the damping of nonlinear cold plasma oscillations is a topic of considerable fundamental interest since it is the simplest nonlinear collective irreversible phenomenon characterizing the plasma state. It also has wide applications to a number of problems of current interest such as particle acceleration by wakefields, beat waves created by intense lasers or particle beams, the fast ignitor concept in inertial fusion, where relativistically intense coupled electromagnetic-plasma wave modes propagate deep into overdense plasmas to create a "hot spark," and a number of other astrophysical and laboratory device-based plasma experiments, where intense plasma oscillations are generated. The conventional thinking about the physics of this interaction is well illustrated by the exact solution for nonlinear one-dimensional cold plasma fluid equations with infinitely massive ions. These exact solutions may be obtained by transforming to Lagrangian coordinates as shown in [1–3] or using stream functions [4]. The exact solution shows that coherent oscillations at the plasma frequency ω_p are maintained indefinitely over the region of initial excitation, provided the normalized amplitude of the initial density perturbation A ($\equiv \frac{\delta n}{n}$) is kept below 0.5. For $A > 0.5$, one expects and observes wave breaking and fine scale mixing of various parts of the oscillation [1]. Mathematically, the electron number density blows up at $A = 0.5$; this is because the Jacobian of transformation from Eulerian to Lagrangian coordinates goes to zero as $A \to 0.5$ and the transformation is no longer unique. Physically, this is equivalent to the crossing of electron trajectories which leads to multistream motions and wave breaking as discussed in [1]. Studies of wave breaking and phase mixing damping are based on numerical simulations.

The above description is adequate when the background positive species are infinitely massive ($\frac{m_-}{m_+} \equiv \Delta \to 0$) and are uniformly distributed. If the background is inhomogeneous, then as was shown by Dawson [1], cold plasma oscillations phase mix away in a time scale $t \sim \frac{\pi}{2(d\omega_p/dx)X}$, at arbitrarily low amplitudes. For a sinusoidal distribution of background species, such a phenomenon in the form of mode coupling of a long wavelength mode to short wavelength modes was observed by Kaw *et al.* [5]. They found that the time scale in which energy goes from long wavelength mode to short wavelength mode is $t \sim \frac{2}{\epsilon \omega_{p0}}$, where "$\epsilon$" is the amplitude of the background inhomogeneity. The exact solution for the cold plasma oscillations in a fixed sinusoidal background was given by Infeld *et al.* [6] who described phase mixing in terms of electron density burst.

In this paper we show that the phenomenon of phase mixing will also occur in a homogeneous plasma at arbitrarily low amplitudes, provided the background positive species are allowed to move ($\Delta \neq 0$). This is because the background species respond to ponderomotive forces either directly or through low frequency self-consistent fields and thereby acquire inhomogeneities in space. Such an effect has been observed in electron positron plasmas ($\Delta = 1$) by Stewart [7]. In plasmas with finite temperature, it is well known that plasma waves dig cavities by ponderomotive forces and get trapped in them; this is the physics of strong turbulence of Langmuir waves as elucidated by Zakharov [8] and leads to envelope soliton formation in one dimension and collapse phenomenon in 2D and 3D. In a cold plasma, stationary states cannot form even in 1D because there is no thermal pressure effect to counterbalance the ponderomotive forces. The result is that the density cavities being dug by plasma oscillations have an amplitude which increases secularly in time. Similarly, the response of plasma oscillations to the presence of density cavities is also different from that of the Zakharov problem. In the Zakharov problem, the thermally dispersive plasma waves get trapped in density cavities forming localized wave packets. Here, the inhomogeneity of the cold plasma (because of the self-consistently generated perturbation) causes different parts of the plasma oscillation to oscillate at different frequencies [1,5,6] resulting in intense phase mixing of plasma oscillations. Thus we physically expect that if the background species are allowed to move and get redistributed into inhomogeneous clumps of density, the phase mixing damping of cold plasma oscillations should come in at any amplitude and is not restricted to waves with $A > 0.5$. It may be emphasized here that, for many applications involving the interaction of superintense laser beams with plasmas (such as particle acceleration by wakefields, penetration into overdense plasmas, etc.), the cold plasma limit considered by us is more relevant than the Zakharov description, because typically the plasma wave intensities are such that $|E|^2/4\pi nT \gg 1$.

In this paper we carry out particle simulations for elucidating the physics of phase mixing damping of nonlinear cold plasma oscillations in an arbitrary mass ratio plasma (Δ arbitrary). We also present a perturbation theoretic analysis to give a quantitative estimate of the phase mixing time for moderate amplitude oscillations and compare it with simulation.

We start with the cold plasma equations, viz., the continuity equations and the equations of motion for the two species and the Poisson equation. We introduce new variables $V, v, \delta n_d$, and δn_s defined as $V = v_+ + v_-$, $v = v_+ - v_-$, $\delta n_d = \delta n_+ - \delta n_- = n_+ - n_-$, and $\delta n_s = \delta n_+ + \delta n_- = n_+ + n_- - 2$ to write the cold plasma equations in the form,

$$\partial_t \delta n_d + \partial_x \left[v + \frac{V \delta n_d + v \delta n_s}{2} \right] = 0 \,, \tag{1}$$

$$\partial_t \delta n_s + \partial_x \left[V + \frac{V \delta n_s + v \delta n_d}{2} \right] = 0 \,, \tag{2}$$

$$\partial_t V + \partial_x \left(\frac{V^2 + v^2}{4} \right) = -(1 - \Delta) E \,, \tag{3}$$

$$\partial_t v + \partial_x \left(\frac{V v}{2} \right) = (1 + \Delta) E \,, \tag{4}$$

$$\partial_x E = \delta n_d \,. \tag{5}$$

Note that we have used the normalizations $n_\pm \to n_\pm/n_0$, $x \to kx$, $t \to \omega_{p-} t$, $v_\pm \to v_\pm/\omega_{p-} k^{-1}$, $E \to E/(4\pi n_0 e k^{-1})$, with $\omega_{p-}^2 = 4\pi n_0 e^2/m_-$ and $\Delta = m_-/m_+$.

Using $n_-(x,0) = 1 + \delta \cos kx$, $n_+(x,0) = 1$, and $v_\pm(x,0) = 0$, as initial conditions, the solutions of the linearized equations are

$$\delta n_d^{(1)} = A \cos kx \cos \omega_p t \,, \tag{6}$$

$$E^{(1)} = \frac{A}{k} \sin kx \cos \omega_p t \,, \tag{7}$$

$$\delta n_s^{(1)} = \frac{1 - \Delta}{1 + \Delta} A \cos kx (1 - \cos \omega_p t) - A \cos kx \,, \tag{8}$$

$$V^{(1)} = -\frac{1 - \Delta}{k \omega_p} A \sin kx \sin \omega_p t \,, \tag{9}$$

$$v^{(1)} = \frac{1 + \Delta}{k \omega_p} A \sin kx \sin \omega_p t \,, \tag{10}$$

where $A = -\delta$ and $\omega_p^2 = 1 + \Delta$. At this level of approximation, the solutions show coherent oscillations at the plasma frequency ω_p. Both of the species oscillate with the same frequency which is independent of position.

In the second order, the solutions are expressed as

$$\delta n_d^{(2)} = -A^2 \cos 2kx \left[\frac{1 - \Delta}{1 + \Delta} \left(\frac{1}{2} + \frac{1}{4} \omega_p t \sin \omega_p t + \frac{1}{2} \cos 2\omega_p t - \cos \omega_p t \right) - \frac{1}{4} \omega_p t \sin \omega_p t \right], \tag{11}$$

$$\delta E^{(2)} = -\frac{A^2}{2k} \sin 2kx \left[\frac{1 - \Delta}{1 + \Delta} \left(\frac{1}{2} + \frac{1}{4} \omega_p t \sin \omega_p t + \frac{1}{2} \cos 2\omega_p t - \cos \omega_p t \right) - \frac{1}{4} \omega_p t \sin \omega_p t \right], \tag{12}$$

$$\begin{aligned} \delta n_s^{(2)} = \frac{A^2}{2} \cos 2kx \Bigg[& \frac{\Delta}{1 + \Delta} t^2 - \frac{\Delta(1 - \Delta)}{(1 + \Delta)^2} \omega_p t \sin \omega_p t - \frac{3}{8} (1 - \cos 2\omega_p t) - \left(\frac{1 - \Delta}{1 + \Delta} \right)^2 \\ & \times \left(2 \cos \omega_p t - \frac{5}{8} \cos 2\omega_p t - \frac{11}{8} \right) \Bigg], \end{aligned} \tag{13}$$

$$\begin{aligned} V^{(2)} = -\frac{A^2}{2k} \sin 2kx \Bigg[& \frac{\Delta}{1 + \Delta} t + \frac{\omega_p}{2} \left(\frac{1 - \Delta}{1 + \Delta} \right)^2 \left(\frac{1 - 3\Delta}{2(1 - \Delta)} \omega_p t \cos \omega_p t + \frac{7 - 5\Delta}{2(1 - \Delta)} \sin \omega_p t - \frac{5}{4} \sin 2\omega_p t \right) \\ & - \frac{1}{8} \omega_p \sin 2\omega_p t \Bigg], \end{aligned} \tag{14}$$

$$v^{(2)} = -\frac{A^2 \omega_p}{8k} \sin 2kx \left[\sin \omega_p t - \omega_p t \cos \omega_p t - \frac{1 - \Delta}{1 + \Delta} (2 \sin 2\omega_p t - 3 \sin \omega_p t - \omega_p t \cos \omega_p t) \right]. \tag{15}$$

The second order solutions clearly exhibit the generation of the second harmonic in space and time as well as the bunching of plasma particles in space. Both of these features are also evident in the solution of Kaw *et al.* [5] and Infeld *et al.* [6], but in contrast to their work, where the background ion density was kept fixed in time, here the density of the plasma particles self-consistently changes with time as $\sim t^2$, as seen in the expression for $\delta n_s^{(2)}$. Because of the variation of plasma density with time, the phase mixing of an initial coherent oscillation happens much faster in this case. To make an estimate of the phase mixing time, consider the charge density equation (δn_d in this case). The equation for δn_d correct up to third order stands as

$$\partial_{tt}\delta n_d + \omega_p^2\left[1 + \frac{1}{2}(\delta n_s^{(1)} + \delta n_s^{(2)})\right]\delta n_d \approx 0\,. \quad (16)$$

In the above equation, if we neglect the second order term, then we essentially get the same phase mixing time as in Ref. [5] modified by a factor which depends on Δ. Now taking only the leading order secular terms from the expressions of $\delta n_s^{(1)}$ and $\delta n_s^{(2)}$ (there are no secular terms in $\delta n_s^{(1)}$) we get

$$\partial_{tt}\delta n_d + \omega_p^2\left[1 + \frac{A^2 t^2 \Delta}{4\omega_p^2}\cos 2kx\right]\delta n_d \approx 0\,. \quad (17)$$

Using the initial conditions $\delta n_d = A\cos kx$ and $\partial_t \delta n_d = 0$ the WKB solution of the above equation is

$$\delta n_d \approx A\cos kx \sum_{n=-\infty}^{n=\infty} \cos\left(\omega_p t + \frac{n\pi}{2} - 2nkx\right) \times J_n\left(\frac{A^2 t^3 \Delta}{24\sqrt{1+\Delta}}\right). \quad (18)$$

The above expression clearly shows that the energy which was initially in the primary wave at mode k goes into higher and higher harmonics as time progresses. This can be interpreted as damping of the primary wave due to mode coupling to higher and higher modes. Microscopically, as the plasma particles oscillate at the local plasma frequency, they gradually go out of phase and eventually the initial coherence is lost. Because of the generation of higher and higher harmonics with time, the charge density becomes more and more spiky and as a result the electric field gradients become more and more steep. This does not go on indefinitely. In reality, the density peaks get limited by thermal effects with the Landau damping of high k modes by resonant particles coming into the picture. This process takes energy from the high k modes and puts it on the particles, thereby raising their temperature, which in turn limits the density peaks by exerting a pressure gradient. The time scale in which the initial coherence is lost (or the phase mixing time) can be seen from Eq. (18) as $\omega_{p-}t_{\text{mix}}$ scale as $\sim[A^2\Delta/(24\sqrt{1+\Delta})]^{-1/3}$. It shows that, only for the ideal case $\Delta = 0.0$ (infinitely massive ions), phase mixing time is infinity, i.e., the initial coherence is maintained indefinitely [2,3]. For an actual electron-ion plasma, Δ, although small, is finite and hence plasma oscillations in it phase mix away at arbitrarily small amplitudes and in a time scale dictated by the amplitude of the initial perturbation.

Now we present results from a 1D particle-in-cell simulation which confirms our scaling of phase mixing time. For numerical simulation, we have used a 1D model with periodic boundary conditions and have followed 5120 electrons and as many positively charged particles (the plasma taken as a whole is neutral) in their own self-consistent fields. The particles are initially at rest and the system is set into motion by giving a density perturbation of the form $n_- = 1 + \delta\cos kx$ to the electrons. In the simulation, we follow the time development of various modes of charge density (δn_d). To compare with our theoretical model, we rewrite Eq. (18) as

$$\delta n_d = \frac{A}{2}\sum_{n=-\infty}^{n=\infty} J_n[\alpha(t)]\left\{\cos\left(\omega_p t + \frac{n\pi}{2}\right)[\cos(2n+1)kx + \cos(2n-1)kx] + \sin\left(\omega_p t + \frac{n\pi}{2}\right) \times [\sin(2n+1)kx + \sin(2n-1)kx]\right\}, \quad (19)$$

where $\alpha(t) = (A^2 t^3/24)(\Delta/\sqrt{1+\Delta})$. The amplitude of the first Fourier mode can be seen from the above equation as

$$|\delta n_d|_{n=1} = \frac{A}{2}[J_0^2(\alpha(t)) + J_1^2(\alpha(t))]^{1/2}. \quad (20)$$

It is clear from Eq. (19) that up to the order of approximation considered, there are no even number modes in the system. Figure 1 shows temporal variation of $|\delta n_d|_{n=1}$ for $\Delta = 1.0$ and $A = 0.05$. The dotted curve is the simulation result and the solid line shows our expression (20) for the envelope of the oscillations. It is clear from the figure that our approximate expression (20) captures the early evolution of the plasma quite well.

Figure 2 shows the variation of $\tau_{\text{mix}} = \omega_p t_{\text{mix}}$ with Δ for a fixed $A = 0.1$ [curve (1)] and with A for a fixed $\Delta = 0.01$ [curve (2)]. These curves clearly confirm our formula for phase mixing time.

In conclusion, we have demonstrated that nonlinear plasma oscillations in a cold homogeneous plasma phase mix away at arbitrarily low amplitudes. This is because during the course of motion the plasma particles respond to ponderomotive forces, acquiring inhomogeneity and thereby making the plasma frequency a function of space. As a result, electrons at different locations

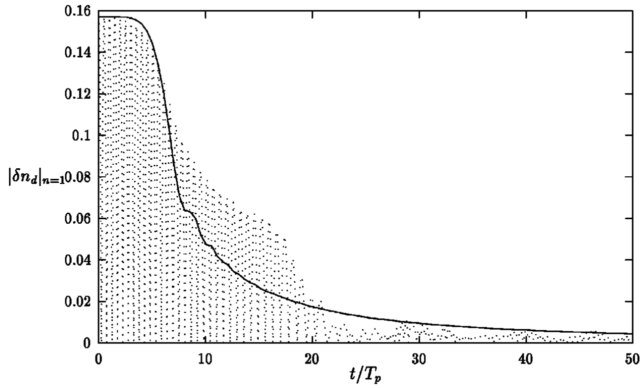

FIG. 1. $|\delta n_d|_{n=1}$ vs t/T_P for $\Delta = 1.0$ and $A = 0.05$.

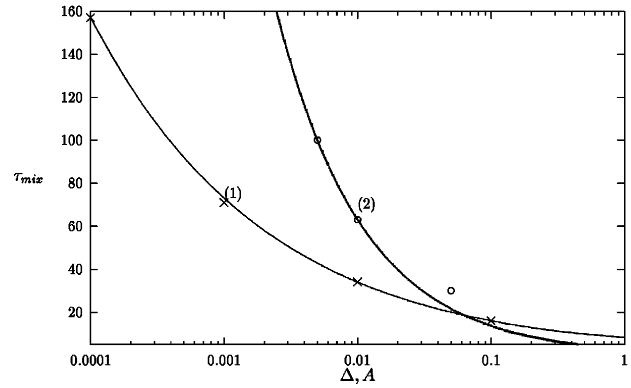

FIG. 2. τ_{mix} vs Δ and A.

oscillate with different (local) plasma frequencies and the imposed plasma wave loses coherence. The formation of density clumps can also be seen from the Zakharov equations [8] for a warm electron-ion plasma. According to Zakharov, the slow variation (in the ion time scale) of the background density in the presence of a high frequency oscillation is governed by $\partial_{tt}\delta n_s - T\partial_{xx}\delta n_s = \partial_{xx}|E|^2$. In the limit when the thermal term balances the ponderomotive force term (i.e., $\frac{\delta n_s}{n_0} \approx -\frac{|E|^2}{T}$), we get caviton solutions in 1D which are unstable to transverse perturbations. In the other limit, when $\frac{|E|^2}{T} \gg 1$, it is the $\partial_{tt}\delta n_s$ term which dominates, and the Zakharov equation shows the density rising as $\sim t^2$. This is the same scaling as obtained by us using a perturbative approach. The density inhomogeneities thus created lead to phase mixing and the collapse of cavitons. From this we infer that a cold 1D plasma exhibits a "Langmuir collapse" phenomenon similar to what is seen in a warm plasma in two or three dimensions. The time scale of collapse is of the order $\sim[(A^2/2)(\Delta/\sqrt{1+\Delta})]^{-1/3}$ plasma periods. Recent experiments on plasma acceleration by laser wakefields have shown [9] that wave breaking of excited plasma oscillations plays a major role in the final acceleration process; similar physics is likely to be important in the fast ignitor concept of laser fusion. We expect the processes discussed in the present paper to play some role in such experiments. It should be noted that the simulation results presented in this Letter are nonrelativistic. For many experimental situations the jitter velocity of electrons is relativistic and we expect the mass ratio Δ to be replaced by $\Delta_{\text{eff}} \approx m_{\text{eff}}/m_i$, where $m_{\text{eff}}/m_i \gg 1$. Under these conditions, the phase mixing effects considered by us should become more important. Such investigations are in progress and will be presented elsewhere.

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