Quantum Computation over Continuous Variables

Seth Lloyd
MIT Department of Mechanical Engineering, MIT 3-160, Cambridge, Massachusetts 02139

Samuel L. Braunstein
SEECS, University of Wales, Bangor LL57 1UT, United Kingdom
(Received 27 October 1998)

This paper provides necessary and sufficient conditions for constructing a universal quantum computer over continuous variables. As an example, it is shown how a universal quantum computer for the amplitudes of the electromagnetic field might be constructed using simple linear devices such as beam splitters and phase shifters, together with squeezers and nonlinear devices such as Kerr-effect fibers and atoms in optical cavities. Such a device could in principle perform "quantum floating point" computations. Problems involving noise, finite precision, and error correction are discussed. [S0031-9007(99)08418-5]

PACS numbers: 03.67.Lx, 46.05.+b

Quantum computation has traditionally concerned itself with the manipulation of discrete systems such as quantum bits, or "qubits" [1,2]. Many quantum variables, such as position and momentum or the amplitudes of electromagnetic fields, are continuous. Although noise and finite precision make precise manipulations of continuous variables intrinsically more difficult than the manipulation of discrete variables, because of the recent developments in quantum error correction [3–5] and quantum teleportation [6,7] of continuous quantum variables, it is worthwhile addressing the question of quantum computation over continuous variables.

At first it might seem that quantum computation over continuous variables is an ill-defined concept. First consider quantum computation over discrete variables. A universal quantum computer over discrete variables such as qubits can be defined to be a device that can by local operations perform any desired unitary transformation over those variables [1,2,8]. More precisely, a universal quantum computer applies "local" operations that effect only a few variables at a time (such operations are called quantum logic gates): By repeated application of such local operations it can effect any unitary transformation over a finite number of those variables to any desired degree of precision. Now consider the continuous case. Since an arbitrary unitary transformation over even a single continuous variable requires an infinite number of parameters to define, it typically cannot be approximated by any finite number of continuous quantum operations such as, for example, the application of beam splitters, phase shifters, squeezers, and nonlinear devices to modes of the electromagnetic field. It is possible, however, to define a notion of universal quantum computation over continuous variables for various subclasses of transformations, such as those that correspond to Hamiltonians that are polynomial functions of the operators corresponding to the continuous variables: A set of continuous quantum operations will be termed universal for a particular set of transformations if one can by a finite number of applications of the operations approach arbitrarily closely to any transformation in the set.

This paper provides necessary and sufficient conditions for universal quantum computation over continuous variables for transformations that are polynomial in those variables. Such a continuous quantum computer is shown to be capable in principle of performing arithmetical manipulations of continuous variables in a "quantum floating point" computation. In principle, a continuous quantum computer could perform tasks that a discrete quantum computer cannot. In practice, noise and finite precision make quantum floating point operations, like their classical counterparts, effectively discrete. A quantum computer that uses continuous variables cannot therefore perform a task that a discrete quantum computer cannot. However, continuous quantum computers may still be able to perform some tasks *more efficiently* than their discrete counterparts. The results derived here apply to any collection of continuous variables, including phonons, photons, Josephson junction circuits, Bose-Einstein condensates, etc. To be concrete, as results are derived they will be expressed both in terms of abstract continuous variables and in the familiar context of quadrature amplitudes of the electromagnetic field.

Consider a single continuous variable corresponding to an operator X. Let P be the conjugate variable: $[X, P] = i$. For example, X and P could correspond to quadrature amplitudes of a mode of the electromagnetic field (the quadrature amplitudes are the real and imaginary parts of the complex electric field). First investigate the problem of constructing Hamiltonians that correspond to arbitrary polynomials of X and P. It is clearly necessary that one be able to apply the Hamiltonians $\pm X$ and $\pm P$ themselves. In the Heisenberg picture, applying

 0031-9007/99/82(8)/1784(4)$15.00

a Hamiltonian H gives a time evolution for operators $\dot{A} = i[H, A]$, so that $A(t) = e^{iHt}A(0)e^{-iHt}$. Accordingly, applying the Hamiltonian X for time t takes $X \to X$, $P \to P - t$, and applying P for time t takes $X \to X + t$, $P \to P$: The Hamiltonians X and P have the effect of shifting the conjugate variable by a constant. In the case of the electromagnetic field, these Hamiltonians correspond to linear displacements or translations of the quadrature amplitudes.

A simple geometric construction allows one to determine what Hamiltonian transformations can be constructed by the repeated application of operations from some set. Apply the Hamiltonian A for time δt, followed by B, $-A$, $-B$, each for the same time. Since

$$e^{iA\delta t}e^{iB\delta t}e^{-iA\delta t}e^{-iB\delta t} = e^{(AB-BA)\delta t^2} + O(\delta t^3), \quad (1)$$

in the limit that $\delta t \to 0$, the result is the same as if one had applied the Hamiltonian $i[A, B]$ for time δt^2. In general, if one can apply a set of Hamiltonians $\{\pm H_i\}$, one can construct any Hamiltonian that is a linear combination of Hamiltonians of the form $\pm i[H_i, H_j]$, $\pm[H_i, [H_j, H_k]]$, etc. [9–13], and no other Hamiltonians. That is, one can construct the Hamiltonians in the algebra generated from the original set by commutation. This key point, originally derived in the context of quantum control and discrete quantum logic, makes it relatively straightforward to determine the set of Hamiltonians that can be constructed from simpler operations.

Now apply this result to the continuous variables introduced above. Since $[X, P] = i$, the application of the translations $\pm X$ and $\pm P$ for short periods of time clearly allows the construction of any Hamiltonian $aX + bP + c$ that is linear in X and P; this is all that it allows. To construct more complicated Hamiltonians one must also be able to perform operations that are higher order polynomials in X and P. Suppose now that one can apply the quadratic Hamiltonian $H = (X^2 + P^2)/2$. Since $\dot{P} = i[H, P] = X$, $\dot{X} = i[H, X] = -P$, application of this Hamiltonian for time t takes $X \to \cos tX - \sin tP$, $P \to \cos tP + \sin tX$. If X and P are quadrature amplitudes of a mode of the electromagnetic field, then H is just the Hamiltonian of the mode (with frequency $\omega = 1$) and corresponds to a phase shifter. Hamiltonians of this form can be enacted by letting the system evolve on its own or by inserting artificial phase delays. Note that since e^{iHt} is periodic with period $1/4\pi$, one can effectively apply $-H$ for a time δt by applying H for a time $4\pi - \delta t$. The simple commutation relations between H, X, and P imply that the addition of $\pm H$ to the set of operations that can be applied allows the construction of Hamiltonians of the form $aH + bX + cP + d$.

Suppose that in addition to translations and phase shifts one can apply the quadratic Hamiltonian $\pm S = \pm(XP + PX)/2$. S has the effect $\dot{X} = i[S, X] = X$, $\dot{P} = i[S, P] = -P$, i.e., applying $+S$ takes $X \to e^t X$, $P \to e^{-t}P$: S "stretches" X and "squeezes" P by some amount. Similarly $-S$ squeezes X and stretches P. In the case of the electromagnetic field, S corresponds to a squeezer operating in the linear regime. It can easily be verified that $[H, S] = i(X^2 - P^2)$. Looking at the algebra generated from X, P, H, and S by commutation, one sees that translations, phase shifts, and squeezers allow the construction of any Hamiltonian that is quadratic in X and P, and of no Hamiltonian of higher order.

To construct higher order Hamiltonians, nonlinear operations are required. One such operation is the "Kerr" Hamiltonian $H^2 = (X^2 + P^2)^2$, corresponding to a χ^3 process in nonlinear optics. This higher order Hamiltonian has the key feature that whereas commuting the previous Hamiltonians, X, P, H, and S with some polynomial in X and P resulted in a polynomial with the same or lower order, commuting H^2 with a polynomial in X and P typically *increases* its order. By evaluating a few commutators, e.g., $[H^2, X] = i(X^2P + PX^2 + 2P^3)/2$, $[H^2, P] = -i(P^2X + XP^2 + 2X^3)/2$, $[X, [H^2, S]] = P^3$, $[P, [H^2, S]] = X^3$ one sees that the algebra generated by X, P, H, S, and H^2 by commutation includes all third order polynomials in X and P. A simple inductive proof now shows that one can construct Hamiltonians that are arbitrary Hermitian polynomials in any order of X and P. Suppose that one can construct any polynomial of order M or less, where M is of degree at least 3. Then since $[P^3, P^mX^n] = iP^{m+2}X^{n-1} +$ lower order terms, and $[X^3, P^mX^n] = iP^{m-1}X^{n+2} +$ lower order terms, one can by judicious commutation of X^3 and P^3 with monomials of order M construct any monomial of order $M + 1$. Since any polynomial of order $M + 1$ can be constructed from monomials of order $M + 1$ and lower, by applying linear operations and a single nonlinear operation a finite number of times one can construct polynomials of arbitrary order in X and P to any desired degree of accuracy. Comparison with similar results for the discrete case [14] shows that the number of operations required grows as a small polynomial in the order of the polynomial to be created, the accuracy to which that polynomial is to be enacted, and the time over which it is to be applied.

The use of the Kerr Hamiltonian H^2 was not essential: Any higher order Hamiltonian will do the trick. Note that commutation of a polynomial in X and P with X and P themselves (which have order 1) always reduces the order of the polynomial by at least 1, commutation with H and S (which have order 2) never increases the order, and commutation with a polynomial of order 3 or higher typically increases the order by at least 1. Judicious commutation of X, P, H, and S with an applied Hamiltonian of order 3 or higher therefore allows the construction of arbitrary Hermitian polynomials of any order in X and P.

The above set of results shows that simple linear operations, together with a single nonlinear operation,

allow one to construct arbitrary polynomial Hamiltonian transformations of a single quantum variable. Let us now turn to more than one variable, e.g., the case of an interferometer in which many modes of the electromagnetic field interact. Suppose now that there are many variables $\{X_i, P_i\}$ on each of which the simple single-variable operations described above can be performed. Now let the variables interact with each other. For simplicity, we assume that we can apply interaction Hamiltonians of the form $\pm B_{ij} = \pm(P_i X_j - X_i P_j)$: A more complicated interaction Hamiltonian can always be used to generate interactions of this form by combining it with single-variable operations. Since $\dot{X}_i = i[B_{ij}, X_i] = X_j$, $\dot{X}_j = i[B_{ij}, X_j] = -X_i$, $\dot{P}_i = i[B_{ij}, P_i] = P_j$, $\dot{P}_j = i[B_{ij}, P_j] = -P_i$, this operation has the effect of taking $A_i \rightarrow \cos tA_i + \sin tA_j$, $A_j \rightarrow \cos tA_j - \sin tA_i$, for $A_i = X_i, P_i$, $A_j = X_j, P_j$. For the electromagnetic field, B_{ij} functions as a beam splitter, linearly mixing together the two modes i and j. By repeatedly taking commutators of B_{ij} with polynomials in X_i, P_i, for different i, it can be easily seen by the same algebraic arguments as above that it is possible to build up arbitrary Hermitian polynomials in $\{X_i, P_i\}$.

This concludes the derivation of the main result: Simple linear operations on continuous variables, together with any nonlinear operation and any interaction between variables suffice to enact to an arbitrary degree of accuracy Hamiltonian operators that are arbitrary polynomials the set of continuous variables. In the case of modes of the electromagnetic field, linear operations such as translations, phase shifts, squeezers, and beam splitters, combined with some nonlinear operation such as a Kerr nonlinearity, allow one to perform arbitrary polynomial transformations on those modes. Note that in contrast to the case of qubits, in which a nonlinear coupling between qubits is required to perform universal quantum computation, in the continuous case only *single variable* nonlinearities are required, along with linear couplings between the variables.

In analog with information over classical continuous variables, which is measured in units of "nats" (1 nat = $\log_2 e$ bits), the unit of continuous quantum information will be called the "qunat." Two continuous variables in the pure state $|\psi\rangle_{12}$ possess $-\mathrm{tr}\,\rho_1 \ln \rho_1$ qunats of entanglement, where $\rho_1 = \mathrm{tr}_2 |\psi\rangle_{12}\langle\psi|$. For two squeezed vacua (squeezed by an amount e^{-r}) entangled using a beam splitter as in Refs. [5–7] the entropy so computed from the approximate Einstein-Podolsky-Rosen state is given by

$$S(\rho) = (1 + \bar{n}) \ln(1 + \bar{n}) - \bar{n} \ln \bar{n} \text{ qunats} \qquad (2)$$

with $\bar{n} = e^r \sinh r$. For example, $e^{2r} = 10$ gives 10 dB of squeezing in power, corresponding to $r = 1.151\,29$. By Eq. (2), two continuous variables entangled using a 10 dB squeezer then possess 2.607 77 qunats of shared, continuous quantum information, equivalent to 3.762 21 qubits of discrete quantum information. This is comparable to the degree of entanglement currently available using ion-trap quantum computers.

Quantum computation over continuous variables can be thought of as the systematic creation and manipulation of qunats. Universal quantum computation for polynomial transformations of continuous variables effectively allows one to perform quantum floating point manipulations on those variables. For example, it is clearly possible using linear operations alone to take the inputs X_1, X_2 and to map them to X_1, $aX_1 + bX_2 + c$. Similarly, application of the three-variable Hamiltonian $X_1 X_2 P_3$ takes $X_1 \rightarrow X_1$, $X_2 \rightarrow X_2$, $X_3 \rightarrow X_3 + X_1 X_2 t$: That is, this operation allows one to multiply X_1 and X_2 and place the result in the "register" X_3. A wide variety of quantum floating point operations are possible. Any polynomial transformation of the continuous variables is clearly possible, as is any transformation that can be infinitesimally represented by a convergent power series. Just as classical computation over continuous variables in principle allows one to solve problems more rapidly than is possible digitally [14], it is interesting to speculate that quantum computation over continuous variables might in principle allow the solution of problems more rapidly than is possible using a "conventional," discrete quantum computer. Continuous variable computation has its own set of problems that might be sped up by the application of continuous quantum computation: For example, such a continuous quantum computer might be used to investigate continuous *NP*-complete problems such as the four-feasibility problem, that is, the problem of deciding whether or not a real degree 4 polynomial in n variables has a zero [15]. In practice, of course, due to finite precision a continuous quantum computer will effectively be able to solve the same set of problems that a conventional discrete quantum computer can, although it may be able to perform some operations more efficiently.

The ability to create and manipulate qunats depends crucially on the strength of squeezing and of the nonlinearities that one can apply. 10 dB squeezers (6 dB after attenuation in the measurement apparatus) currently exist [16]. High Q cavity quantum electrodynamics can supply a strong Kerr effect in a relatively lossless context, and quantum logic gates constructed for qubits could be used to provide the nonlinearity for continuous quantum variables as well [17]. Here the fact that only single-mode nonlinearities are required for universal quantum computation simplifies the problem of effecting continuous quantum logic. Nonetheless, the difficulty of performing repeated nonlinear operations in a coherent and loss-free manner is likely to limit the possibilities for quantum computation over the amplitudes of the electromagnetic field. Vibrational modes of ions in traps or excitations of a Bose-Einstein condensate might provide the long-lived, lossless states required for quantum computation over continuous variables.

Noise poses a difficult problem for quantum computation [18–20], and continuous variables are more susceptible to noise than discrete variables. Since an uncountably infinite number of things can go wrong with a continuous variable, it might at first seem that continuous error correction routines would require infinite redundancy. In fact, continuous quantum error correction routines exist and require no greater redundancy than conventional routines [3–5]. Such routines are capable of correcting for noise and decoherence in principle: In practice, measurement noise, losses, and the lack of perfect squeezing will lead to imperfect error correction [5]. Surprisingly, continuous quantum error correction routines are in some sense easier to enact than discrete quantum error correction routines, in that the continuous routines can be implemented using only *linear* operations together with classical feedback [5]. The relative simplicity of such routines suggests that robust, fault-tolerant quantum computation may in principle be possible for continuous quantum variables as well as for qubits. (A scheme for quantum computation is fault tolerant if quantum computations can be carried out even in the presence of noise and errors [21,22]. A fault-tolerant scheme that allows for arbitrarily long quantum computations to be carried out is said to be robust [23].) If this is indeed the case then quantum computation over continuous variables, despite its intrinsic difficulties, may be an experimentally viable form of quantum information processing. Continuous variables might be used to simulate continuous quantum systems such as quantum field theories. Even in the absence of fault tolerance, the large bandwidths available to continuous quantum computation make it potentially useful for quantum communications and cryptography [24].

S. L. thanks H. Haus and H. J. Kimble for useful discussions. This work was supported by DARPA under the QUIC initiative.

[1] D. DiVincenzo, Science **270**, 255 (1995).
[2] S. Lloyd, Sci. Am. **273**, No. 4, 140 (1995).
[3] S. Lloyd and J. J.-E. Slotine, Phys. Rev. Lett. **80**, 4088 (1998).
[4] S. L. Braunstein, Phys. Rev. Lett. **80**, 4084 (1998).
[5] S. L. Braunstein, Nature (London) **394**, 47 (1998).
[6] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[7] A. Furusawa *et al.*, Science **282**, 706 (1998).
[8] This definition of quantum computation corresponds to the normal "circuit" definition of quantum computation as in, e.g., D. Deutsch, Proc. R. Soc. London A **425**, 73 (1989); A. C.-C. Yao, in *Proceedings of the 36th Annual Symposium on Foundations of Computer Science*, edited by S. Goldwasser (IEEE Computer Society, Los Alamitos, CA, 1995), pp. 352–361. The works of M. Reck *et al.*, Phys. Rev. Lett. **73**, 58 (1994); N. J. Cerf, C. Adami, and P. G. Kwiat, Phys. Rev. A **57**, R1477 (1998), showing how to perform arbitrary unitary operators using only linear devices such as beam splitters, though of considerable interest and potential practical importance, does not constitute quantum computation by the usual definition. Reck *et al.* and Cerf *et al.* propose performing arbitrary unitary operations on N variables not by acting on the variables themselves but by expanding the information in the variables into an interferometer with $O(2^N)$ arms and acting in this exponentially larger space. Local operations on the original variables correspond to highly nonlocal operations in this "unary" representation: To flip a single bit requires one to act on half $[O(2^{N-1})]$ of the arms of the interferometer. Actually to perform quantum computation on qubits using an interferometer requires nonlinear operations as detailed in Y. Yamamoto, M. Kitagawa, and K. Igeta, in *Proceedings of the 3rd Asia-Pacific Physics Conference*, edited by Y. W. Chan, A. F. Leung, C. N. Yang, and K. Young (World Scientific, Singapore, 1988), pp. 779–799; G. J. Milburn, Phys. Rev. Lett. **62**, 2124 (1989).
[9] G. M. Huang, T. J. Tarn, and J. W. Clark, J. Math. Phys. (N.Y.) **24**, 2608–2618 (1983).
[10] *Differential Geometric Control Theory*, edited by R. W. Brockett, R. S. Millman, and H. J. Sussman (Birkhauser, Boston, 1983); *Nonholonomic Motion Planning*, edited by Z. Li and J. F. Canney (Kluwer Academic, Boston, 1993).
[11] V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz, and A. Peirce, Phys. Rev. A **51**, 960–966 (1995).
[12] S. Lloyd, Phys. Rev. Lett. **75**, 346–349 (1995).
[13] D. Deutsch, A. Barenco, and A. Ekert, Proc. R. Soc. London A **449**, 669–677 (1995).
[14] S. Lloyd, Science **273**, 1073 (1996).
[15] L. Blum, M. Shub, and S. Smale, Bull. Am. Math. Soc. **21**, 1–46 (1989).
[16] L. A. Wu *et al.*, Phys. Rev. Lett. **57**, 2520 (1986).
[17] Q. A. Turchette *et al.*, Phys. Rev. Lett. **75**, 4710–4713 (1995).
[18] R. Landauer, Nature (London) **335**, 779–784 (1988).
[19] R. Landauer, Phys. Lett. A **217**, 188–193 (1996).
[20] R. Landauer, Philos. Trans. R. Soc. London A **335**, 367–376 (1995)
[21] P. Shor, *Proceedings of the 37th Annual Symposium on the Foundations of Computer Science* (IEEE Computer Society Press, Los Alamitos, 1996), pp. 56–65.
[22] D. P. DiVincenzo and P. W. Shor, Phys. Rev. Lett. **77**, 3260–3263 (1996).
[23] R. Laflamme, M. Knill, and W. H. Zurek, Science **279**, 342 (1998); D. Aharanov and M. Ben-Or, quant-ph/9611025; J. Preskill, Proc. R. Soc. London A **454**, 385 (1998).
[24] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India* (IEEE Press, New York, 1984), pp. 175–179; A. K. Ekert *et al.*, Phys. Rev. Lett. **69**, 1293 (1992); P. D. Townsend, J. G. Rarity, and P. R. Tapster, Electron. Lett. **29**, 1291 (1993); R. J. Hughes *et al.*, in *Advances in Cryptology: Proceedings of Crypto 96* (Springer-Verlag, New York, 1997), pp. 329–343; A. Muller *et al.*, Appl. Phys. Lett. **70**, 793 (1997).