Partially Dual Variables in SU(2) Yang-Mills Theory

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We propose a reformulation of SU(2) Yang-Mills theory in terms of new variables, appropriate for describing the theory in its infrared limit. These variables suggest a dual picture of the Yang-Mills theory where the short distance limit describes asymptotically free, massless point gluons and the large distance limit describes extended, massive knotlike solitons. [S0031-9007(99)08490-2]

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In the high energy limit the Yang-Mills theory is asymptotically free and can be solved perturbatively. It describes the interactions of massless gluons which correspond to the transverse polarizations of the gauge field A_μ [1].

At low energies the Yang-Mills theory becomes strongly coupled. Perturbative techniques fail and nonperturbative methods must be developed. To this, substantial efforts have been devoted, but numerical lattice approaches still remain the most viable tool to effectively explore the low energy theory. Despite our lacking theoretical understanding of the low energy Yang-Mills theory, we expect that it exhibits color confinement with ensuing mass gap. The physical spectrum is supposed to describe massive composites of A_μ such as glueballs. When quarks are introduced the gauge field should form stringlike flux tubes which confine quarks inside hadrons.

In the present Letter we propose an approach to investigate the SU(2) Yang-Mills theory in the infrared limit. Our proposal is motivated by the qualitative picture developed in particular by 't Hooft [2] and Polyakov [2], who asserted that the ultraviolet and infrared limits of a Yang-Mills theory represent different phases, with color confinement due to a dual Meissner effect in a condensate of magnetic monopoles. This picture suggests that even though the gauge field A_μ is the proper order parameter for describing the theory in its ultraviolet limit, in the infrared limit with monopole condensation some other order parameter could become more adequate. Naturally we expect that such a change of variables may also imply a certain need to reformulate the Yang-Mills action.

In the high energy limit the theory is described by the standard Yang-Mills action

$$S = \frac{1}{g^2}\int dx\,\mathrm{Tr}\,F^2. \qquad (1)$$

This is the *unique* Lorentz and gauge invariant local action which is renormalizable in four dimensions and admits a Hamiltonian interpretation which identifies the transverse polarizations of A_μ as the physical fields present in the ultraviolet limit.

In the following we shall propose new variables for describing the infrared limit of a four dimensional SU(2) Yang-Mills theory. We argue that, instead of A_μ, in this limit the appropriate order parameter involves a three component vector $n^a(x)$ $(a = 1,2,3)$ with unit length $\mathbf{n}\cdot\mathbf{n} = 1$. When combined with standard Wilsonian renormalization group arguments, this suggests that the action contains [3,4]

$$S = \int dx\, m^2(\partial_\mu \mathbf{n})^2 + \frac{1}{e^2}(\mathbf{n}, \mathbf{dn}\times\mathbf{dn})^2. \qquad (2)$$

Here m is a mass scale and e is a dimensionless coupling constant: This is the *unique* local and Lorentz-invariant action for the unit vector $\mathbf{n}$ which is at most quadratic in time derivatives so that it admits a Hamiltonian interpretation and involves *all* such terms that are either relevant or marginal in the infrared limit.

Observe that the action (2) can be related to the SU(2) Skyrme model, restricted to a sphere S^2. However, the topological features of these two models are quite different.

We note that in four dimensions the action (2) fails to be perturbatively renormalizable in the ultraviolet. But since it is expected to describe the physical excitations of a SU(2) Yang-Mills theory in the low energy strong coupling limit, lack of perturbative renormalizability should not pose a problem provided that we can interpret (2) adequately. Indeed, we have recently established [5] that in 3 + 1 dimensions the classical action (2) describes stable knotlike solitons. This suggests that a proper route to its quantization should be based on the investigation of the quantum mechanical properties of these solitons.

From the point of view of a Yang-Mills theory the presence of knotlike solitons is actually quite appealing. It is natural to relate these solitons with the stringlike flux tubes that we expect to be present in the infrared spectrum of a Yang-Mills theory, to provide the confining force

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between two quarks. In the absence of quarks such flux tubes may still be present. They now close on themselves in knotted, stable solitonic configurations which are natural candidates for describing glueballs. In this manner we arrive at a dual picture of the Yang-Mills theory, with the high energy limit described by massless and pointlike transverse polarizations of A_μ and the low energy limit described by massive solitonic flux tubes which close on themselves in stable knotlike configurations.

We shall now proceed to present our novel parametrization of the connection A_μ^a: In the picture developed in particular by 't Hooft [2], confinement is viewed as a dual Meissner effect in a condensate of magnetic monopoles. In a SU(2) Yang-Mills theory the relevant magnetic monopole is the (singular) Wu-Yang configuration [6]

$$A_i^a = \epsilon_{aik} \frac{x_k}{r^2}, \qquad A_0^a = 0. \tag{3}$$

In order to describe a condensate of these monopoles, we need to properly extend (3) by introducing a smooth field for the corresponding order parameter. A natural ansatz for extending (3) into a condensate is

$$A_i^a = \epsilon_{abc}\partial_i n^b n^c \equiv \mathbf{dn} \times \mathbf{n}, \tag{4}$$

with $\mathbf{n}$ a three component unit vector field that describes the condensate. It reproduces (3) when we specify to the singular "hedgehog" configuration

$$n = \mathbf{x}/r. \tag{5}$$

The unit vector $\mathbf{n}$ describes two independent field variables. Since a *gauge fixed* four dimensional SU(2) connection $\mathbf{A}_\mu$ describes six polarization degrees of freedom, we need to extend the parametrization (4) by four additional polarizations. In order to search for a natural extension, we first observe that under an infinitesimal gauge transformation

$$\delta A_\mu^a = \nabla_\mu^{ab}\varepsilon^b = \partial_\mu \varepsilon^a + \epsilon^{acb} A_\mu^c \varepsilon^b,$$

which is parametrized by the Lie algebra element $\varepsilon^a(x) = \varepsilon(x) n^a(x)$, (4) fails to remain form invariant. But if we improve (4) into

$$\mathbf{A}_\mu = C_\mu \mathbf{n} + \mathbf{dn} \times \mathbf{n}, \tag{6}$$

where $C_\mu(x)$ is a vector field which transforms as an Abelian connection

$$C_\mu \to C_\mu + \partial_\mu \varepsilon, \tag{7}$$

the functional form of the configuration (6) remains intact under this gauge transformation.

The functional form (6) of SU(2) connections has been previously studied in particular by Cho [7], as a consistent truncation of the full four dimensional connection A_μ^a. He was interested in identifying those field degrees of freedom in A_μ^a which are relevant for describing the Abelian dominance, a concept that originates from [2] and is expected to be relevant for color confinement.

The Abelian gauge invariance (7) implies that (6) describes four field components, corresponding to the two transverse polarizations of the U(1) connection C_μ and the two independent components of $\mathbf{n}$. In order to extend (6) so that it describes all six field components of an arbitrary connection $\mathbf{A}_\mu$, we consider an arbitrary finite gauge transformation of a generic connection A_μ. With

$$U(x) = \exp\left\{ i\, \frac{1}{2}\, \alpha \mathbf{n} \cdot \tau \right\},$$

the SU(2) group element that determines this gauge transformation, we find for the gauge transformation of an arbitrary connection A_μ^a

$$\begin{aligned} \mathbf{A}^U = {} & [(\mathbf{A}, \mathbf{n}) + \mathbf{d}\alpha]\mathbf{n} + \mathbf{dn} \times \mathbf{n} \\ & + \sin\alpha(\mathbf{dn} + \mathbf{A} \times \mathbf{n}) \\ & - \cos\alpha(\mathbf{dn} + \mathbf{A} \times \mathbf{n}) \times \mathbf{n}. \end{aligned} \tag{8}$$

From this we conclude that a generic connection $\mathbf{A}_\mu$ should have the functional form

$$\mathbf{A}_\mu = C_\mu \mathbf{n} + \mathbf{dn} \times \mathbf{n} + \varphi \mathbf{B}_\mu + \mathbf{B}_\mu \times \mathbf{n}, \tag{9}$$

where φ is a scalar field and B_μ^a is an orthogonal SU(2) valued vector, $\mathbf{n} \cdot \mathbf{B}_\mu = 0$ for all μ. Since the number of independent field components carried by a *four* dimensional SU(2) connection is six, the orthogonal field $\mathbf{B}_\mu$ should describe only a single component, and we can select it to be proportional to $\mathbf{dn}$. This yields the following ansatz for parametrizing a generic *four* dimensional connection,

$$\mathbf{A}_\mu = C_\mu \mathbf{n} + \mathbf{dn} \times \mathbf{n} + \rho \mathbf{dn} + \sigma \mathbf{dn} \times \mathbf{n}. \tag{10}$$

Notice that we have here separated the second and fourth terms on the right-hand side, even though these terms are linearly dependent. The reason for this separation is that it allows us to combine the scalars ρ and σ into a complex field

$$\phi = \rho + i\sigma \tag{11}$$

with the property that under a SU(2) gauge transformation generated by $\alpha^a = \alpha \cdot \mathbf{n}$ the functional form of (10) remains intact, with the multiplet (C_μ, ϕ) transforming like the field multiplet in the Abelian Higgs model.

If we again specify to the singular hedgehog configuration (5), we find that (10) incorporates the most general cylindrically symmetric SU(2) connection [8]. Indeed, our main proposal is the *completeness* of the parametrization (10) in four dimensions. For this we substitute (10) to the classical Yang-Mills action (1) and derive equations of motion obtained by varying the component fields $(\mathbf{n}, C_\mu, \phi)$. For completeness these equations should reproduce the original Yang-Mills equations $\nabla_\mu \mathbf{F}_{\mu\nu} = 0$, obtained by *first* varying w.r.t. $\mathbf{A}_\mu$ in (1) and then substituting (10).

If we introduce the U(1) covariant derivative

$$D_\mu \phi = \partial_\mu \phi + iC_\mu \phi = \partial_\mu \rho - C_\mu \sigma + i(\partial_\mu \sigma + C_\mu \rho) = D_\mu \rho + iD_\mu \sigma \,, \tag{12}$$

we find

$$\mathbf{F}_{\mu\nu} = \mathbf{n}\{G_{\mu\nu} - [1 - (\rho^2 + \sigma^2)]H_{\mu\nu}\} + (D_\mu \rho \partial_\nu \mathbf{n} - D_\nu \rho \partial_\mu \mathbf{n}) + (D_\mu \sigma \partial_\nu \mathbf{n} \times \mathbf{n} - D_\nu \sigma \partial_\mu \mathbf{n} \times \mathbf{n}) \,, \tag{13}$$

where

$$G_{\mu\nu} = \partial_\nu C_\mu - \partial_\mu C_\nu \,,$$
$$H_{\mu\nu} = (\mathbf{n}, \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \,.$$

When we substitute (13) into the Yang-Mills action (1) we get

$$S = \frac{1}{g^2} \int dx \, \{\mathbf{n}[G_{\mu\nu} - (1 - [\rho^2 + \sigma^2])H_{\mu\nu}] + (D_\mu \rho \partial_\nu \mathbf{n} - D_\nu \rho \partial_\mu \mathbf{n}) + (D_\mu \sigma \partial_\nu \mathbf{n} \times \mathbf{n} - D_\nu \sigma \partial_\mu \mathbf{n} \times \mathbf{n})\}^2, \tag{14}$$

and when we perform the variations w.r.t. $(C_\mu, \phi, \mathbf{n})$ we get

$$\mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \,,$$
$$\partial_\nu \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \,,$$
$$\partial_\nu \mathbf{n} \times \mathbf{n} \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \,,$$
$$(D_\nu \rho + D_\nu \sigma \cdot \mathbf{n} \times) \cdot \nabla_\mu \mathbf{F}_{\mu\nu} = 0 \,,$$

which are all proportional to the Yang-Mills equation $\nabla_\mu \mathbf{F}_{\mu\nu} = 0$, evaluated at the field (10). But the U(1) invariance (7) implies that in four dimensions only six of these equations can be independent. These equations coincide with the six independent second order equations that we obtain when we first vary the action (1) w.r.t. the full connection A_μ^a and then substitute for the parametrization (10). Thus we assert that the parametrization (10) is indeed complete. [We remind one that the variation of (1) w.r.t. A_μ^a yields twelve equations, but the three A_0^a are Lagrange multipliers and three of the equations are first order, corresponding to Gauss law in the Hamiltonian approach. Consequently, in four dimensional SU(2) Yang-Mills theory there are only six independent second order equations.]

We observe that the second term in (2) is already present in (14). Furthermore, if in the original Yang-Mills equation $\nabla_\mu \mathbf{F}_{\mu\nu} = 0$ we substitute (10) with $C_\mu = \phi = 0$, we find that the result also coincides with the equation we obtain by varying the second term in (2) with respect to $\mathbf{n}$. The first term in (2) involves a mass scale and it is absent in (1),(14). Indeed, there is no way to introduce a mass scale in four dimensional Yang-Mills by employing ultraviolet renormalizable, local, Lorentz, and gauge invariant functionals of A_μ. However, when we represent the Yang-Mills action using the component field (9),(10), standard Wilsonian renormalization group arguments suggest that in the infrared limit the first term in (2) should also be included: It is a relevant operator in the infrared; hence it should emerge when we account for fluctuations in a gradient expansion. This can be verified explicitly, by integrating over the Abelian Higgs multiplet in (14) employing, e.g., a heat-kernel expansion [9]. Alternatively, we simply average over the scalar field $\phi = \rho + i\sigma$ in (14) to the effect

$$\langle |\partial_\lambda \phi|^2 \eta_{\mu\nu} - \partial_\mu \phi^* \partial_\nu \phi \rangle = m^2 \eta_{\mu\nu} \,. \tag{15}$$

As a consequence we conclude that the full action (2) is contained in a gradient expansion of the effective action for the order parameter $\mathbf{n}$.

Obviously the full effective action for the order parameter $\mathbf{n}$ obtained by integrating over the complete set of fields in the parametrization (9) will also contain various additional functionals of $\mathbf{n}$ besides the two terms that appear in (2). However, (2) is *unique* in the sense that it contains *all* such infrared relevant and marginal, local Lorentz invariant operators of $\mathbf{n}$ which are at most quadratic in time derivatives, as is necessary for a Hamiltonian interpretation. In this sense (2) is the *unique* action to describe the low energy limit of a SU(2) Yang-Mills theory, in the confining phase where magnetic monopoles condense. The results of [5] then suggest that at low energies the physical states of the Yang-Mills theory are knotlike solitons of the monopole condensate, and it becomes natural to view these configurations as candidates for describing glueballs.

Notice that the present interpretation of (2) is entirely analogous to the common point of view to consider (1) as the fundamental action for the high energy Yang-Mills theory, even though, e.g., a gradient expansion of the lattice Yang-Mills action involves higher derivative terms which all become irrelevant in the continuum (short-distance) limit where the lattice spacing tends to zero.

Besides the order parameter $\mathbf{n}$ which is appropriate for describing the phase with monopole condensation, we have also found that the Abelian Higgs multiplet (C_μ, ϕ) naturally appears in the parametrization of four dimensional connections. Elimination of $\mathbf{n}$ in (14) then produces an effective action for the Abelian Higgs multiplet which comprises a natural order parameter for describing the SU(2) theory in a "Higgs phase," also considered in [2]. Indeed, since we have the spontaneously broken Higgs self-coupling present in (14),

$$V(\phi) = \langle H_{\mu\nu}^2 \rangle (1 - [\rho^2 + \sigma^2])^2 \sim \lambda(1 - |\phi|^2)^2,$$

we can expect the corresponding effective action to support Nielsen-Olesen-type vortices as infinite energy line solitons. In a sense, these Abelian Higgs variables can be viewed as dual to the vector field **n** in the expansion (10). It would be of interest to study further the properties of this Higgs phase.

In conclusion, we have derived a novel parametrization of the SU(2) Yang-Mills field, appropriate for describing the theory in its infrared limit. As an application, we have argued that (2) is the unique action for describing SU(2) Yang-Mills theory at low energies, consistent with various natural first principles. In particular, the first term in (2) should be included since it is relevant in the infrared limit. This term introduces a mass gap and the ensuing action supports knotlike configurations as stable solitons. Our parametrization then suggests a dual picture of the Yang-Mills theory where the high energy limit describes massless pointlike gluons and the infrared limit describes massive knotted solitons, consistent with the commonly accepted picture of color confinement.

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