Topological Spectral Correlations in 2D Disordered Systems

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(Received 17 September 1998)

It is shown that the tail in the two-level spectral correlation function $R(s)$ for 2D disordered systems depends on the system geometry and the boundary conditions. In particular, for closed surfaces (with no boundary), $R(s) = -\chi/(6\pi^2\beta s^2)$, where $\beta = 1$, 2 or 4 for the orthogonal, unitary, and symplectic ensembles, respectively, and $\chi = 2(1-p)$ is the Euler characteristic of the surface with p "handles" (holes). The result is valid for $g \ll s \ll g^2$ for $\beta = 1$, 4 and for $g \ll s \ll g^3$ for $\beta = 2$, where $g \gg 1$ is the dimensionless conductance. [S0031-9007(98)08123-X]

PACS numbers: 73.20.Fz, 72.15.Rn

Spectral correlations in complex and disordered systems are closely related to the basic symmetries of a system, the nature of its dynamics, and the structure of the corresponding eigenstates. It is remarkable that there are few universal spectral statistics which all the generic complex and disordered systems obey in the thermodynamic (TD) limit. According to the well-known Bohigas-Giannoni-Schmidt conjecture [1], a generic chaotic system is described by the Wigner-Dyson (WD) spectral statistics which follows from one of the three universal Gaussian ensembles of random matrices [2]. In contrast, all of the generic integrable systems obey the Poisson spectral statistics. Depending on the strength of disorder, spectral statistics of a d-dimensional disordered system of noninteracting particles also flow to one of the above universal statistics as the system size L tends to infinity. For strong disorder and for low-dimensional systems ($d = 1, 2$) where all states are localized, the spectral statistics in the TD limit are Poissonian. For weakly disordered 3D systems where all states are extended, the spectral statistics are identical to the WD [3] in the TD limit.

However, the disordered systems with $d > 2$ are known to undergo the Anderson localization transition at a certain critical disorder. At this point the localization (correlation) length ξ diverges and the spectral statistics should be independent of the system size L, provided that the energy difference $\omega = s\Delta$ is measured in units of the mean level spacing $\Delta \propto L^{-d}$. Thus at the Anderson transition there exists a special fixed point, the critical spectral statistics (CSS) [4,5].

An amazing property of some CSS found recently [6] is the sensitivity to boundary conditions and a shape of a sample. It will follow from the analysis below that the basic properties of the CSS take their "canonical" form for the *periodic boundary conditions* (PBC). Other boundary conditions induce some characteristic features in the CSS which depend on the properties of the boundary.

In order to study the qualitative role of the system geometry for the CSS we consider the system which shows all of the typical features of the critical system and allows for a rigorous analysis. This is a 2D weakly disordered electron system. The small parameter which allows for a rigorous treatment is the inverse dimensionless conductance $g^{-1} = \Delta/E_c \ll 1$, where $\hbar/E_c = t_D = L^2/D$ is the diffusion time and D is the diffusion coefficient. For a not too large system size $L \ll \xi$, where $\xi = l\exp[g]$ or $\xi = l\exp[g^2]$ is the localization radius in the orthogonal or unitary ensemble, respectively, one can neglect the localization effects and consider the dimensionless conductance g as independent of the size L. At such conditions the behavior of the 2D system is similar to the one in the true critical point.

The two-level correlation function $R(s)$ in such 2D systems has been considered in Ref. [7] for the case of the periodic boundary conditions *only*. It has been shown that the two-level correlation function $R(s)$ is *exponentially small* at $s \gg g$ if one neglects the weak-localization effects. This means that the so-called Altshuler-Shklovskii [8] tail $R(s) = C_d s^{-2+d/2}$ is *absent* in 2D systems. This statement is true [5] for a generic critical system.

In the present Letter we consider $R(s)$ at $s \gg g$ for an arbitrary 2D surface S of the area A_S. We will show that, in the case where the surface has a nontransparent boundary, the two-level correlation function has a power-law tail $R(s) \sim g^{-1/2}s^{-3/2}$. Moreover, we will show that even in the absence of boundaries (e.g., for the sphere) the correlation function $R(s)$ still has a power-law tail $R(s) = Cs^{-2}$. This tail at $s \gg g$ is of the same form as the universal WD tail for $1 \ll s \ll g$. However, the coefficient C depends on the *topology* of the surface and is equal to zero only for the torus topology which is equivalent to the PBC.

The two-level correlation function is defined as

$$R(\omega) = \frac{1}{\rho^2}\left[\left\langle \rho\left(E + \frac{\omega}{2}\right)\rho\left(E - \frac{\omega}{2}\right)\right\rangle - 1\right], \quad (1)$$

where $\langle\cdots\rangle$ denote averaging over all realizations, $\rho = \langle\rho(E)\rangle$ is the average one-electron density of states (DOS), and $\rho(E)$ is the DOS for a particular realization of disorder:

$$\rho(E) = S^{-1} \sum_n \delta(E - E_n), \qquad (2)$$

where E_n are exact one-electron energy levels. Since we are interested in the energy scale $\omega = s\Delta$ which vanishes in the TD limit, the average DOS is considered *energy independent*.

It is remarkable that in the limit $g \gg 1$ and $s \gg 1$ the spectral correlation function $R(s)$ of the *quantum* problem can be expressed [8] entirely in terms of the eigenvalues $\varepsilon_\mu \Delta/\hbar$ of the *classical* diffusion problem with the Neumann boundary conditions which correspond to the nontransparent boundary:

$$R(s) = \frac{1}{\pi^2 \beta} \operatorname{Re} \sum_\mu \frac{1}{(\varepsilon_\mu - is)^2}, \qquad (3)$$

where $\beta = 1$, 2, or 4 for the orthogonal, unitary, and symplectic ensembles, respectively [2]. Note that the dimensionless conductance $g = \varepsilon_1 - \varepsilon_0$ is just the gap between the lowest ($\varepsilon_0 = 0$) and the first nonzero eigenvalues.

This means that, independently of the details of the electron conduction band structure and the short-range correlated random potential, at $g \gg 1$ there exists a region of s where the electron level correlations depend only on the spectrum ε_μ of the Laplace-Beltrami operator $\Delta_{\mathcal{G}}$ on a curved surface:

$$\Delta_{\mathcal{G}} = \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial}{\partial x^i} \left(\sqrt{\mathcal{G}}\, \mathcal{G}^{ij} \frac{\partial}{\partial x^j} \right), \qquad (4)$$

where $\mathcal{G}^{ij} = [\hat{\mathcal{G}}^{-1}]_{ij}$, $\mathcal{G} = \det \hat{\mathcal{G}}$, and $\hat{\mathcal{G}}$ is a metric tensor on the surface.

Equation (3) can be rewritten in the following form [9]:

$$K(t) = \int_{-\infty}^{+\infty} \frac{ds}{2\pi} e^{-its} R(s) = \frac{1}{2\pi^2 \beta} |t| p(|t|), \qquad (5)$$

if one introduces the classical return probability $P(t, \mathbf{r})$ to the point $\mathbf{r}$,

$$P(t, \mathbf{r}) = \sum_\mu \exp[-t\varepsilon_\mu] [\Phi_\mu(\mathbf{r})]^2, \qquad (6)$$

and the averaged return probability,

$$p(t) = \int_S d\mathbf{r} P(t, \mathbf{r}) = \sum_\mu \exp[-t\varepsilon_\mu], \qquad (7)$$

where $\Phi_\mu(\mathbf{r})$ is an eigenfunction of the diffusion (Laplace-Beltrami) operator and $t = T/t_H = T\Delta/\hbar$ is time in units of the Heisenberg time t_H.

Equation (5) sets the relationship between the tail in $R(s)$ at $s \gg 1$ and the small-time behavior of the return probability $p(t)$ at $t \ll 1$. There are two regions of t with different behavior of $p(t)$: the diffusion region $t \ll 1/g$ and the ergodic region $1/g \ll t \ll 1$. In the ergodic region $t\varepsilon_\mu \gg 1$ for all $\mu \neq 0$, so that only the zero mode $\varepsilon_0 = 0$ contributes to Eqs. (6) and (7). In this region $p(t) = 1$ is independent of t, and one obtains the universal WD result $R(s) = -\frac{1}{\pi^2 \beta} s^{-2}$.

In the diffusion region the main contribution to the return probability is given by the continuous approximation in Eq. (7). In this approximation $p(t)$ is well known to be proportional to $1/t$ in 2D systems, thus giving rise to a constant $K(t)$. It is the property of all of the critical systems that the main contribution to $K(t)$ is independent of t for sufficiently small times [10,11]. This means that the tail in $R(s)$ for $s \gg g$ is determined in critical systems entirely by the *corrections* to the return probability. There are two types of such corrections. One source of corrections is the *weak-localization* effects considered in Ref. [7] or the finite-size corrections to scaling considered in Ref. [10]. In 2D systems they lead to

$$R_{WL}(s) \sim \begin{cases} \mp g^{-2} s^{-1}, & \beta = 1,4. \\ -g^{-3} s^{-1}, & \beta = 2. \end{cases} \qquad (8)$$

Another source of corrections which has not been considered so far is the corrections to the continuous approximation in Eq. (7) that is the difference between the sum and the corresponding integral. This correction is sensitive to the *boundary conditions* and *topology* of the surface.

The sensitivity of $p(t)$ to the boundary conditions originates from the $\mathbf{r}$ dependence of the return probability $P(t, \mathbf{r})$. If the point $\mathbf{r}$ is at a sufficiently small distance $x \ll L$ from a smooth boundary, the $\mathbf{r}$-dependent correction $\delta G(t, \mathbf{r}, \mathbf{r}) = \delta P_b(t, \mathbf{r})$ to the Green's function of the diffusion equation with the Neumann boundary conditions, is given by the "image source": $\delta P_b(t, \mathbf{r}) = (4\pi DT)^{-1} \exp[-x^2/DT]$. Then, assuming the smooth boundary of the length $\mathcal{L}$ one obtains [12] for the corresponding correction to the *averaged* return probability $p(t)$,

$$\delta p_b(t) = \int_0^\infty \delta P_b(t, \mathbf{r}) \mathcal{L}\, dx = \frac{\mathcal{L}}{8\sqrt{\pi DT}}. \qquad (9)$$

The boundary-induced tail in $R(s)$ is found immediately from Eqs. (5) and (9):

$$R_b(s) = -\frac{1}{8\beta (\pi g_b)^{1/2} (2\pi s)^{3/2}}, \qquad (10)$$

where $g_b = D/\mathcal{L}^2 \Delta$. Note that for $\mathcal{L} \sim L$ we have $g_b \sim g$.

Comparing Eqs. (8) and (10), one can see that the boundary-induced power-law tail in $R(s)$ is larger than the localization-induced tail if $g \ll s \ll g^3$ for $\beta = 1,4$ and $g \ll s \ll g^5$ for $\beta = 2$.

If the boundary is absent at all [$R_b(s) \equiv 0$], there is still a correction to the continuous approximation in Eq. (7). This correction is *universal* and depends only on the *topology* of the surface. In order to understand how such a topological correction may arise at times T which are small compared with the diffusion time, let us consider the return probability $P(t, \mathbf{r})$. For times $T \ll t_D$ ($t \ll 1/g$) the diffusing electron probes only the vicinity of the starting point $\mathbf{r}$. Therefore, the correction to the return probability $P(t, \mathbf{r})$ may depend only on the *local* curvature. We will show below [see Eq. (18)] that it is proportional to the Gauss curvature $K_{\mathbf{r}} = k_{\mathbf{r}}^{(1)} k_{\mathbf{r}}^{(2)}$ in a point $\mathbf{r}$, where $k^{(1)}$ and $k^{(2)}$ are the principal curvatures. Information about the topology of a surface arises because of the integration over all positions of the starting point $\mathbf{r}$ in Eq. (7). Indeed, according to the Gauss-Bonnet theorem [13], the integral of $K_{\mathbf{r}}$ over a surface S is related to the Euler characteristic $\chi(S)$ of the surface:

$$\int_S K_{\mathbf{r}} d\sigma = 2\pi\chi(S). \tag{11}$$

The quantity $\chi(S) = V + F - E$ is related to the number of vertices V, edges E, and faces F of the surface triangulation and depends on the *connectivity* of the surface, i.e., on the number p of "handles" (holes):

$$\chi(S) = 2(1 - p), \tag{12}$$

where $p = 0$ for a sphere and $p = 1$ for a torus.

In the simplest case of a sphere where each eigenvalue $\varepsilon_\mu = l(l + 1)$ of the Laplace-Beltrami operator Eq. (4) is $(2l + 1)$ times degenerate, the topological correction to $p(t)$ at $tg \ll 1$ can be obtained by a straightforward summation in Eq. (7) using the well known [14] formula $\sum_0^\infty f(l) \approx \int_{-1/2}^\infty f(l)dl + \frac{1}{24}f'(-1/2)$:

$$p(t) = \sum_{l=0}^\infty (2l + 1)\exp[-tgl(l + 1)] \approx \frac{1}{tg} + \frac{1}{3}. \tag{13}$$

One can see that for a sphere ($p = 0$) the topological correction to $p(t)$ at $tg \ll 1$ is a *universal constant* $\frac{1}{3}$. Then using Eq. (12) and $\delta p(t) \propto \chi(S)$, we conclude that for a generic surface the topological correction reads

$$\delta p_{\mathrm{top}}(t) = \frac{1 - p}{3}. \tag{14}$$

This correction is time independent and one immediately obtains from Eq. (5) a topological tail in $R(s)$:

$$R_{\mathrm{top}}(s) = -\frac{1 - p}{3\pi^2\beta s^2}, \qquad (s \gg g). \tag{15}$$

Comparing this correction with the weak-localization correction Eq. (8), we see that the latter is smaller at $g \ll s \ll g^2$ or $g \ll s \ll g^3$ for $\beta = 1,4$ or $\beta = 2$, respectively. Note that the topological tail has the same form as the universal WD tail. However, the former is valid in the *diffusion* region $s \gg g$ while the latter is valid in the *ergodic* region $s \ll g$.

The result [Eq. (14)] follows from the theory of the Laplacian on Riemannian manifolds (TLRM) [15] and has been used [12] to find corrections to the semiclassical density of states in quantum billiards and resonators [16]. Here we present an elementary derivation of Eq. (14) which does not require the full power of the TLRM.

For an arbitrary nonsingular point $\mathbf{r}$ on a surface, it is always possible to choose the local system of coordinates (ξ_1, ξ_2) such that the inverse metric tensor $\mathcal{G}^{ij}(\xi_1, \xi_2)$ in the vicinity of the origin (point $\mathbf{r}$) takes the form:

$$\mathcal{G}^{11} = \mathcal{G}^{22} = 1, \qquad \mathcal{G}^{12} = \mathcal{G}^{21} = -K_{\mathbf{r}}\xi_1\xi_2, \tag{16}$$

where $K_{\mathbf{r}}$ is the Gaussian curvature at a point $\mathbf{r}$, and higher order terms in $\xi^{1,2}$ are omitted. Then, we obtain for the Laplace-Beltrami operator [Eq. (4)],

$$\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} - K_{\mathbf{r}}\left(\frac{\partial}{\partial\xi_1}\xi_1\xi_2\frac{\partial}{\partial\xi_2} + \frac{\partial}{\partial\xi_2}\xi_1\xi_2\frac{\partial}{\partial\xi_1}\right) \equiv \hat{\Delta} + \hat{V}. \tag{17}$$

As was mentioned above, for small times $T \ll t_D$, the term proportional to the Gaussian curvature $K_{\mathbf{r}}$ can be treated as a perturbation. The return probability $P(t, \mathbf{r})$ is given by the Green's function $G(t, \mathbf{r}, \mathbf{r})$ of the diffusion operator $\partial/\partial\tilde{t} - \Delta_{\mathcal{G}}$, where $\tilde{t} = DT$. Then, all we should do in order to find the correction to the local return probability is to compute the correction to the Green's function $G^{(1)} = G^{(0)}\hat{V}G^{(0)}$, where $G^{(0)} = (4\pi\tilde{t})^{-1}\exp[-(\xi - \xi')^2/4\tilde{t}]$. The result is:

$$P(t, \mathbf{r}) = \frac{1}{4\pi\tilde{t}} + \frac{K_{\mathbf{r}}}{12\pi}. \tag{18}$$

The first term in this expression corresponds to the continuous approximation. The second one is the correction due to the local curvature $K_{\mathbf{r}}$. This very term has been used in deriving Eq. (14).

In conclusion, we have considered the power-law tail in the two-level correlation function $R(s)$ for 2D disordered systems with the diffusion motion of electrons. In the limit of large dimensionless conductance $g \gg 1$, there is an interval of $s \gg g$, where the tail is entirely determined by the geometry of the sample. It consists of the two contributions: a boundary contribution given by Eq. (10) and a topological contribution given by Eq. (15), which are the main results of this Letter. For closed surfaces without a boundary, the tail in $R(s)$ for $s \gg g$ is determined only by topology and is of the same form as the universal Wigner-Dyson tail for $1 \ll s \ll g$. However, the numerical topological prefactor depends on the connectivity of the surface and is negative for single-connected surfaces (spheres), zero for surfaces of the torus topology, and *positive* for surfaces of higher genus.

We are grateful to B. L. Altshuler, E. Bogomolny, I V. Lerner, A. D. Mirlin, G. Montambaux, and I. V. Yurkevich for stimulating discussions. The work was supported by INTAS-RFBR Grant No. 95-0675, RFBR Grant No. 96-02-17133, and the US CRDF Award No. RP1-209 (V. E. K.), and RFBR Grant No. 98-02-16062 (V. I. Y.). V. I. Yudson acknowledges the hospitality of the ICTP in Trieste during the Workshop on Disorder Chaos and Interaction in Mesoscopic Systems, where this work was completed.

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[16] Note that in Ref. [12] the expression for $\delta p_{\mathrm{top}}(t)$, similar to Eq. (14), has an incorrect extra factor of $1/2$ that has been rewritten in a number of further applications to quantum billiards.