Exact Solution of the One-Dimensional Ballistic Aggregation

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An exact expression for the mass distribution $\rho(M;t)$ of the ballistic aggregation model in one dimension is derived in the long time regime. It is shown that it obeys scaling $\rho(M;t) = t^{-4/3}F(M/t^{2/3})$ with a scaling function $F(z) \sim z^{-1/2}$ for $z \ll 1$ and $F(z) \sim \exp(-z^3/12)$ for $z \gg 1$. Applications of these results to Burgers turbulence are discussed. [S0031-9007(99)08466-5]

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Ballistic aggregation is a simple version of a dissipative gas. It provides a dynamical model of nonequilibrium statistical physics in which particles follow the basic laws of mechanics. I thus consider a one-dimensional gas of pointlike massive particles which move freely until they collide. The perfectly inelastic collision of two masses conserves the total mass and momentum, while dissipation occurs as kinetic energy is lost in each collision. One can anticipate the formation of more and more massive while slower and slower aggregates.

This model was introduced by Carnevale, Pomeau, and Young [1] where they conjectured, based on scaling arguments and numerical simulations, an asymptotic scaling regime for the mass distribution $\rho(M;t) = F(M/\langle M\rangle_t)/\langle M\rangle_t^2$. The average mass per aggregate was supposed to grow algebraically with time as $\langle M\rangle_t \sim t^{2/3}$, and the scaling function had a simple universal exponential form $F(z) = \exp(-z)$ independent of the initial conditions. Later, this conjecture was reinforced by Piasecki [2] where he solved the hierarchy of dynamical equations governing the system inside a mean-field approximation scheme.

This system, in its continuous limit, was also studied as a simplified astronomical model for the agglomeration of cosmic dust into macroscopic objects [3]. The ballistic aggregation model, where the aggregates interact only through their collisions, describes the early time behavior of an aggregation model where gravitational interactions are present [4].

It is important to mention the connection between this model and solutions of the Burgers equation. At a very high Reynolds number, the asymptotic solution of the Burgers equation consists of a train of shock waves. The laws of motion which govern the dynamics of these shock waves are found to be equivalent to a ballistic aggregation system (see [5]).

In this Letter, I verify the scaling hypothesis for the mass distribution and find by an exact calculation an explicit form for the scaling function. It happens to be different from the conjectured simple exponential. I remark that such an exact solution is scarce in nonequilibrium systems.

Rather than solving the set of partial differential equations governing the evolution of the system, I exploit the fact that, once the initial state of the system is given, the dynamics is completely deterministic. My approach is thus based on a statistical study of the initial conditions and is largely inspired by the work of Martin and Piasecki [6].

Initially, particles having all the same mass m are regularly placed on a line with the same interparticle distance a. Initial mass density is thus $\rho_0 = m/a$. The initial momenta of the thermalized particles are not correlated and are distributed according to the same Gaussian distribution $\phi_m(p) = \sqrt{\beta/(2\pi m)}\exp[-\beta p^2/(2m)]$. I now choose $\beta = 1/2$ and $\rho_0 = 1/2$ to simplify notations without loss of generality.

I now compute the density distribution $\rho_m(X,M,P,t)$, where $\rho_m(X,M,P,t)\,dM\,dP\,dX$ is the number of aggregates located in $(X, X + dX)$ with momentum in $(P, P + dP)$ and mass in $(M, M + dM)$ at time t.

When the coordinates (X,M,P,t) of an aggregate are given, they uniquely define the number $n = M/m$ as well as the initial positions $x_i = X - Pt/M + m(2i + 1 - n)$, $1 \le i \le n$ of its constituents. A crucial point is that an aggregate, once formed, follows the movement of the center of mass (c.m.) of its constituents, which can be determined from the initial state. I label the location of the c.m. at time t of the r particles located initially at $(j + 1)a, (j + 2)a, \ldots, (j + r)a$ by

$$X_{j+1}^r(t) \equiv \frac{1}{rm}\sum_{i=1}^{r}(mx_{j+i} + tp_{j+i}) = (2j + r + 1)\,\frac{a}{2} + \frac{t}{rm}\sum_{i=1}^{r} p_{j+i}\,. \quad (1)$$

The mass distribution can be determined from the initial conditions and an aggregate of mass $M = mn$ is present at position X and at time t if and only if (i) the aggregate has formed: the c.m. of its leftmost s particles has crossed the c.m. of its rightmost $n - s$ particles for all $s = 1,\ldots,n - 1$ up to time t, leading to $X_{j+1}^s(t) > X_{j+s+1}^{n-s}(t)$ for $1 \ge s \ge n - 1$ and $X_{j+1}^n(t) = X$ with j, an integer; (ii) the aggregate has not been disturbed by other particles: the c.m. of the successive groups of particles not constituting the aggregate has not crossed the c.m. of the aggregate up to time t, thus $X_{j-r+1}^r(t) < X < X_{j+n+1}^r(t)$ for $r \ge 1$.

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One has (see [6] for details)

$$\rho_m(X,M,P,t) = \left\langle \prod_{r=1}^{\infty} \Theta\{X - X_{j-r+1}^r(t)\} \prod_{s=1}^{n-1} \Theta\{X_{j+1}^s(t) - X_{j+s+1}^{n-s}(t)\} \delta\left(P - \sum_{r=1}^{n} p_{j+r}\right) \prod_{r=1}^{\infty} \Theta\{X_{j+n+1}^r(t) - X\} \right\rangle, \tag{2}$$

with Θ being the Heaviside step function, $M = nm$, and where $j = (X - tP/M)/a - (n + 1)/2$ is an integer. The brackets denote the average over the initial distribution of the momenta.

Owing to the uncorrelated initial Gaussian distribution of the momenta, the density distribution (2) factorizes in a product of three functions, one corresponding to the condition (i) above,

$$\begin{aligned} \left\langle \prod_{s=1}^{n-1} \Theta\{X_{j+1}^s(t) - X_{j+s+1}^{n-s}(t)\} \delta\left(P - \sum_{r=1}^{n} p_{j+r}\right) \right\rangle &= \int \prod_{s=1}^{n-1} dP_s\, \phi_m(P_s - P_{s-1}) \Theta\left\{P_s - \frac{sm}{t}\left(t\frac{P}{M} + M - sm\right)\right\} \\ &\quad \times \phi_m(P - P_{n-1}) \\ &= \frac{1}{t^{1/3}} \int \prod_{s=1}^{n-1} dP_s'\, \phi_{m'}(P_s' - P_{s-1}') \Theta\left\{P_s' - sm'\left(\frac{P'}{M'} + M' - sm'\right)\right\} \phi_{m'}(P' - P_{n-1}') = \frac{1}{t^{1/3}} I_{m'}(M', P'), \end{aligned} \tag{3}$$

where we used new variables defined as $P_s = \sum_{i=1}^{s} p_{j+i}$, $1 \le s \le n - 1$ with $P_0 = 0$, and the integration range extends from minus infinity to plus infinity for each P_s. Moreover, we introduced the rescaled variables $M' = M/t^{2/3}$, $P' = P/t^{1/3}$, $m' = m/t^{2/3}$, and $P_i' = P_i/t^{1/3}$. On the other end, Eq. (3) reveals that the function $I_{m'}(M', P')$ is the probability for a Brownian motion in momentum space $P(\tau)$ starting at $P(0) = 0$ to end at $P(nm') = P$ while staying above the discrete points $P(sm') > f(sm')$ for $1 \le s \le n - 1$, with the parabola $f(x) = -x(P'/M' + M' + x)$.

Along the same line, one can easily show that the conditions (ii) above lead to the two contributions in Eq. (2),

$$\left\langle \prod_{s=1}^{\infty} \Theta\{X - X_{j-s+1}^s(t)\} \right\rangle = \int \prod_{s=1}^{\infty} dP_s'\, \phi_{m'}(P_s' - P_{s-1}') \Theta\left\{P_s' + sm'\left(\frac{P'}{M'} + M' + sm'\right)\right\} = J_{m'}\left(M' + \frac{P'}{M'}\right) \tag{4}$$

and

$$\left\langle \prod_{s=1}^{\infty} \Theta\{X_{j+n+1}^s(t) - X\} \right\rangle = J_{m'}\left(M' - \frac{P'}{M'}\right), \tag{5}$$

where the function $J_{m'}(Y)$ is the probability for a Brownian motion starting at $P(0) = 0$ to pass over the discrete points $P(sm') > f_Y(sm')$ for $s \ge 1$, with $f_Y(x) = -x(Y + x)$. The constraints on the Brownian motion are illustrated in Fig. 1.

Using Eqs. (3)–(5), we find the exact scaling form for the distribution (2),

$$\begin{aligned} \rho_m(X,M,P;t) &= \frac{1}{t^{1/3}} \rho_{m'}(M', P'; 1) \\ &= \frac{1}{t^{1/3}} J_{m'}(M' + P'/M') I_{m'}(M', P') \\ &\quad \times J_{m'}(M' - P'/M'). \end{aligned} \tag{6}$$

Note that, due to translational invariance, the mass distribution does not depend on X.

One of the main difficulties of this problem is the discrete nature of the constraints on the Brownian motions. I will derive below an expression for the mass distribution in the limit $m' = m/t^{2/3} \to 0$ (with M' and P' fixed) which is reached either when $t \to \infty$ for a fixed m (asymptotic long time limit) or for any fixed time t when $m \to 0$ (continuous limit). In terms of the Brownian motion introduced above, the space m' between the discrete point barrier shrinks to zero and approaches a continuous barrier which makes the problem tractable analytically. Nevertheless, the functions I and J are identically null for $m' = 0$, which means that the distribution ρ is null for an infinite time. One thus keeps track of the first k points ($M_0 = km'$) of the discrete constraints on the Brownian motion. The double limit $m' \to 0$, and then $M_0 \to 0$ gives an expansion for the functions I and J in power of m', which is equivalent to an asymptotic expansion in t (see [7] for justification and details).

The detailed analytical calculations leading to the explicit forms of the functions I and J are outside the

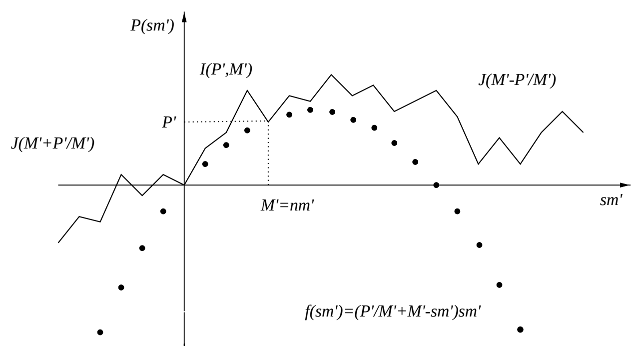

FIG. 1. Constraints on the Brownian motion used in the construction of our solution.

scope of this Letter and will be given elsewhere [7]. One finds the dominant contribution in m'

$$I_{m'}(M', P') = m' e^{-M'^3/12 - P'^2/2M'} \mathcal{I}(M') + \mathcal{O}(m'^{3/2}), \tag{7}$$

with

$$\mathcal{I}(M') = \sum_{k \geq 1} e^{-\omega_k M'}, \tag{8}$$

where $-\omega_k$ ($k \geq 1$) are the zeros of the Airy function [8], which form an infinite and countable set and are located on the negative real axis ($-\omega_1 \simeq -2.33811, -\omega_2 \simeq -4.08795, \ldots$). This function had to be expected in this problem as it is known that it arises in the description of a Brownian motion with a parabolic drift [9].

On the other end, one gets

$$J_{m'}(Y) = \sqrt{m'}\, e^{(Y/2)^3/3} \mathcal{J}(Y) + \mathcal{O}(m'), \tag{9}$$

with

$$\mathcal{J}(Y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \frac{e^{-zY/2}}{\mathrm{Ai}(z)}, \tag{10}$$

where $c > -\omega_1$.

Now, inserting the expression for I and J in Eq. (6), one has

$$\rho(M,P;t) = \frac{m^2}{t^{5/3}} \mathcal{I}\left(\frac{M}{t^{2/3}}\right) \mathcal{J}\left(\frac{M}{t^{2/3}} + \frac{Pt^{1/3}}{M}\right) \times \mathcal{J}\left(\frac{M}{t^{2/3}} - \frac{Pt^{1/3}}{M}\right) + \mathcal{O}\left(\frac{m^{5/2}}{t^2}\right). \tag{11}$$

From this equation one sees that the concentration $c(t)$ of aggregates, the aggregates average mass and momentum, and the mean energy per unit of length $E(t)$ behave, for time $t \gg 1$, as

$$c(t) \sim t^{-2/3}, \qquad \langle M \rangle_t \sim t^{2/3}, \qquad \langle P \rangle_t = 0, \qquad \sqrt{\langle P^2 \rangle_t} \sim t^{1/3}, \qquad E(t) \sim t^{-2/3}. \tag{12}$$

The integration of Eq. (11) over P [7] leads to the mass distribution, which is the main result of this Letter,

$$\rho(M;t) = \frac{m^2}{t^{4/3}} F\left(\frac{M}{t^{2/3}}\right) + \mathcal{O}\left(\frac{m^{5/2}}{t^{5/3}}\right), \tag{13}$$

where one sees that it obeys the expected scaling form with a scaling function,

$$F(M') = 2M' \mathcal{I}(M') \mathcal{H}(M'), \tag{14}$$

where

$$\mathcal{I}(M') = \sum_{k \geq 1} e^{-\omega_k M'}, \qquad \mathcal{H}(M') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \frac{e^{-M'z}}{\mathrm{Ai}^2(z)}, \tag{15}$$

with $c > -\omega_1$.

The scaling function $F(M')$ is plotted in Fig. 2.

One can compare the obtained scaling function with the conjectured one [$F_{\mathrm{conj.}}(M') = \exp(-M')$] [1]. In particular, small and large arguments present strong differences. Indeed, for $M' \ll 1$, one gets $\mathcal{H}(M') = 1 + \mathcal{O}(M')$, while one can estimate $\mathcal{I}(M')$ using the asymptotic properties of the zeros of the Airy function $\omega_k = [(3\pi k)/2]^{2/3} + \mathcal{O}(k^{-1/3})$ and find $\mathcal{I}(M') \sim (2\sqrt{\pi} M'^{3/2})^{-1}$. One thus has

$$F(M') = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{M'}} + \mathcal{O}(\sqrt{M'}), \qquad (M' \ll 1). \tag{16}$$

One can conclude, for example, that the number $N(M_0, t) = \int_0^{M_0} \rho(M;t)\, dM$ of aggregates of small masses $M < M_0 \ll t^{2/3}$ at time t is well underestimated by the conjectured form which leads to $N(M_0, t) \sim M_0/t$, while the exact solution gives $N(M_0, t) \sim \sqrt{M_0}/t$.

For $M' \gg 1$, one can estimate the function $\mathcal{H}(M')$ by the steepest descent method and find $\mathcal{H}(M') \sim \sqrt{\pi} M'^{3/2} \exp(-M'^3/12)$. On the other end, one has $\mathcal{I}(M') \sim \exp(-\omega_1 z)$ leading to

$$F(M') = 2\sqrt{\pi} M'^{5/2} e^{-M'\omega_1 - M'^3/12}, \qquad (M' \gg 1). \tag{17}$$

This is again different from the conjectured function as large masses $M \gg t^{2/3}$ have a much smaller chance to be present in the system than conjectured.

One can compute the collision frequency between two masses M_1 and M_2,

$$\nu_2(M_1, M_2, t) \sim t^{-1} m'^3 (M_1' + M_2') M_1' M_2' \mathcal{I}(M_1') \mathcal{I}(M_2') \times \mathcal{H}(M_1' + M_2'), \tag{18}$$

with $\mathcal{I}, \mathcal{H}$ as above, and where M_1', M_2', and m' are properly rescaled variables. It does not factorize in a

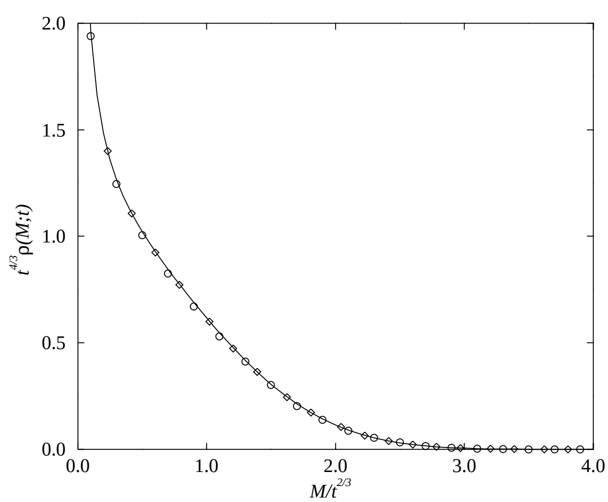

FIG. 2. The exact rescaled mass distribution $t^{4/3}\rho(M;t)$ as a function of $M/t^{2/3}$ for large t (solid line). As an illustration, the results of a numerical simulation of the ballistic aggregation are plotted for different times [$t = 100$ (diamonds) and $t = 1000$ (circles)].

product of functions of M_1 and M_2, respectively, and thus invalidates the assumption on which the mass distribution was computed in [2].

One can inquire about the universality of these results with respect to other initial conditions. Let us first consider a Poissonian distribution of the particle initial positions with an average interparticle distance a. The discrete points over which the Brownian motion should pass in the construction of our solution are still distributed on the same parabola but with irregular spacing. In the long time limit and after rescaling, the spacing between points of average $a' = a/t^{2/3}$ becomes smaller and smaller to be, in first order in m', a continuum. The difference between irregular and regular spacing is thus asymptotically erased and the result, Eq. (13), should be recovered in this case.

A bimodal momentum distribution $\phi(p) = [\delta(p - p_0) + \delta(p + p_0)]/2$ is used as an initial state in [1]. I believe that this should not affect the form of the mass distribution (13) as the random walk initiated by this distribution is well approximated, in the long time limit, by the considered Brownian motion.

One can define a distribution where momenta are initially correlated. In this case, one expects the scaling function to be different, at least for small M' [10].

The Burgers equation, $\partial_t u + u\partial_x u = \nu \partial_x^2 u$, is a simplified form of the Navier-Stokes equation originally introduced as a toy model of hydrodynamic turbulence. In the inviscid limit ($\nu \to 0$), Burgers [5] has shown that, for rough initial conditions and in the long time limit, the solution $u(x,t)$ approaches a sawtooth profile (shock waves). At a given time t, each shock is characterized by its strength ξ/t and its advance velocity η. The law of motion of the shocks is then mapped exactly on the ballistic aggregation model with $\xi = M$ and $\eta = P/M$ [11]. Up to numerical factors coming from the discrete nature of the initial condition of the ballistic aggregation, the exact solutions, Eqs. (11) and (13), solved the distribution $p(\xi, \eta; t)$ and $p(\xi; t)$ of the asymptotic Burgers solutions introduced in [5]. In particular, the behavior of the mass distribution function for small masses, Eq. (16), is compatible with previous numerical results [11]. Note that, in the case of a white noise initial distribution $u(x,0)$, this mapping is valid at all time t and that the asymptotic behaviors of the scaling function Eq. (13) are compatible with the exact bounds found for the Burgers problem [12]. The noisy Burgers equation has attracted many recent studies [13]. In this respect, it would be interesting to clarify in which way a noise can be added to the ballistic aggregation problem. The method exposed in this Letter is, however, based on the deterministic nature of the aggregates dynamics and is of little use if the dynamics is stochastic.

In summary, I found an exact asymptotic solution for the mass distribution of the ballistic aggregation in one dimension. Such an exact solution is not frequent in a nonequilibrium system and has permitted me to verify a scaling hypothesis for this system. While the average mass per aggregate was proved to behave with time as $\langle M \rangle_t \sim t^{2/3}$ for $t \gg 1$, as expected from previous studies, the scaling function is shown here to be different from the conjectured one. This distribution also solves the shock strength distribution of the one-dimensional Burgers equation in the inviscid limit with a white noise initial condition.

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