Information Dynamics of Photorefractive Two-Beam Coupling

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An operator approach to two-beam coupling in photorefractive media for which each beam is composed of information that varies in both space and time reduces the problem to a single differential equation: $d\rho/dz = [[\sigma_0, \rho], \rho]/4$. $[A, B] = AB - BA$ is the commutator of two operators, ρ is a density operator that embodies the state of the optical field, and σ_0 is a coupling operator. A solution exists in closed form for specific cases and is amenable to numerical integration in general. The evolution of ρ is unitary and approaches a block diagonal in a representation that diagonalizes σ_0. [S0031-9007(99)08382-9]

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A coherent pair of laser beams made to intersect in barium titanate or a similar photorefractive material will couple such that one beam will give up its energy to the other, as illustrated in Fig. 1. The process is known as two-beam coupling [1], and it is one of several nonlinear optical interactions in photorefractives that lead to a remarkable variety of effects and devices, such as self- and mutually pumped phase conjugation, oscillation in a resonator, and novelty filters [2]. In addition to having a two-beam coupling gain, photorefractive materials are holographic and have memory. These properties drive the interest in photorefractives for image processing and other information processing applications. Consider a task such as imaging through a turbulent atmosphere or perhaps tracking a bird as it flies among the trees of a forest. These, like many real-time imaging problems, impose both spatial and temporal modulation onto laser beams. What happens when information-bearing beams interact in a photorefractive medium? The conventional approach to nonlinear optical interactions is poorly suited to address questions concerning the information content of the beams. The standard approach decomposes the optical field into modes, usually plane waves, then derives a set of coupled slowly varying amplitude equations to describe how the modes evolve.

This work develops an operator treatment of two-beam coupling interactions. Spatial information encoded on the beams is represented by state vectors in a Hilbert space. As the optical beams propagate through the medium so do the vectors undergo a dynamical evolution in space. We show that a spatially varying Hamiltonian governs this dynamics. Despite the fact that two-beam coupling is fundamentally a nonlinear process, the medium can nevertheless be viewed as a "black box" that performs a unitary transformation on the input vectors. From an information processing standpoint, it is a convenient view that emphasizes the function rather than the mechanism of the process.

Yet the formalism offers more than a convenient viewpoint by revealing aspects of wave mixing in photorefractives that are not so apparent in the standard coupled mode approach. In a series of papers, Hall and collaborators [3–7] identified the symmetries associated with four-wave mixing in optical media by formulating the problem as a dynamics on an SU(2) or other group manifold (depending upon the geometry). They also pointed out that conserved quantities of the dynamics are conveniently imbedded in a matrix formalism, whereas they are imposed constraints in the coupled mode approach. Stojkov and Belic [8] and Stojkov *et al.* [9] independently echoed similar conclusions. They employed a state-vector representation of the fields, as we do. Like these other works, the operator formalism developed here is revealing of the symmetries associated with multiple wave interactions.

In its simplest rendition, two-beam coupling in photorefractive media is described by two interacting plane waves: one at the plus $(+)$ port having amplitude e_+ and one at the minus $(-)$ port having amplitude e_-. An index grating g arises from the interference between the waves, couples the amplitudes, and gives rise to a spatial

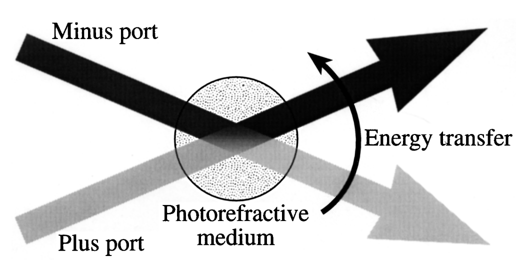

FIG. 1. In simple two-beam coupling, energy is transferred from one beam at the minus port to another at the plus port. An arrow indicates the direction of energy transfer. A typical information processing scenario has both beams spatially and temporally modulated.

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evolution of the fields according to

$$\begin{aligned} e'_+ &= g e_- \,, \\ e'_- &= -g^* e_+ \,, \\ g &= \Gamma e_+ e_-^*/2I \,, \end{aligned} \tag{1}$$

where $'$ indicates d/dz, I is the total intensity, $I = |e_+|^2 + |e_-|^2 = I_+ + I_-$, and Γ is the coupling coefficient. We assume here that the coupling is real, in which case the phases are constant and one usually solves for the intensities,

$$\begin{aligned} I'_+ &= \Gamma I_+ I_-/I \,, \\ I'_- &= -\Gamma I_+ I_-/I \,. \end{aligned} \tag{2}$$

The analytic solution to these equations and their energy transfer characteristics are well known [10–12]: the plus port intensity increases as it derives energy from the minus port beam. An information processing scenario typically requires many modes—perhaps having different carrier frequencies or perhaps appearing at different times. To address this scenario one typically takes both temporal and spatial Fourier transforms of the fields. The equations of motion become

$$\begin{aligned} e'_{+i\omega} &= \sum_j g_{ij} e_{-j\omega} \,, \\ e'_{-j\omega} &= -\sum_i g^\dagger_{ij} e_{+i\omega} \,, \\ g_{ij} &= \Gamma \sum_\omega e_{+i\omega} e^*_{-j\omega}/2I \,, \end{aligned} \tag{3}$$

where $\dagger$ indicates complex-conjugate transpose. Latin indices are used to specify spatial components and Greek indices to specify temporal components. Our operator formalism replaces Eq. (3) with a single equation that embodies the same physics.

We begin the operator formalism by introducing the optical state vector written using the Dirac notation as $|\psi_\omega\rangle$ [13]. In the plane-wave representation, $\sqrt{I_\omega}\,|\psi_\omega\rangle$ has as its components the Fourier amplitudes of the plane waves having carrier frequency ω. For example, the amplitude of the kth spatial Fourier component of the plus port is given by the matrix element, $\sqrt{I_\omega}\,\langle +k \,|\, \psi_\omega\rangle = e_{+k\omega}$. Notice that $|\psi_\omega\rangle$ carries both plus and minus port amplitudes. We explicitly take the intensity factor out of the state vectors so that they are normalized to unity: $\langle\psi_\omega \,|\, \psi_\omega\rangle = 1$. The spatial structures associated with two different temporal components of the field are *not* orthogonal in general $\langle\psi_{\tilde\omega} \,|\, \psi_\omega\rangle \neq 0$. The state vector carries the information in a representation-free manner; one may choose to represent the information in a way that is convenient for the information processing problem at hand. For example, one might choose to represent $|\psi_\omega\rangle$ by the gray-level value of the various pixel elements of a spatial light modulator rather than by its plane-wave components.

It will sometimes be useful to write various state vectors and operators in what we call the experimenter's block representation. In the case of $|\psi_\omega\rangle$,

$$|\psi_\omega\rangle \rightarrow \begin{pmatrix} |\psi_{+\omega}\rangle \\ |\psi_{-\omega}\rangle \end{pmatrix}. \tag{4}$$

The upper state vector represents the beam of the plus port and the lower represents the minus port. Both may have an arbitrary number of modes. The form of the grating amplitude in Eqs. (1) and (3) suggests that it may be useful to construct a density operator from the following sum of temporal components:

$$\rho \equiv \sum_\omega \frac{I_\omega}{I} |\psi_\omega\rangle\langle\psi_\omega| \,. \tag{5}$$

In the experimenter's representation,

$$\rho \rightarrow \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix}. \tag{6}$$

A matrix representation of the density operator has a unit trace and it is Hermitian, $\rho = \rho^\dagger$. Next, we introduce the coupling operator σ_0. In the experimenter's representation,

$$\sigma_0 = \begin{pmatrix} 1_m & 0 \\ 0 & -1_n \end{pmatrix}, \tag{7}$$

where the $1_m, 1_n$ are identity operators of their respective spaces. If, for example, the plus port is exposed to m plane waves and the minus port is exposed to n plane waves, then the upper left identity operator is represented by an $m \times m$ diagonal matrix of 1's, and the lower right is represented by an $n \times n$ diagonal matrix of -1's. From now on we drop the subscripts on the identity operators. Note that the coupling operator has the useful property so that its square is the identity, $\sigma_0^2 = 1$, which means that it is both unitary and Hermitian. We combine the density and coupling operators to introduce a third operator,

$$H_\rho(z) = i[\sigma_0, \rho(z)]/4 \,. \tag{8}$$

Since σ_0 and ρ are both Hermitian, H_ρ is also Hermitian. Equation (3) can now be replaced by an operator counterpart,

$$|\psi'_\omega(z)\rangle = -iH_\rho(z)\,|\psi_\omega(z)\rangle \,, \tag{9}$$

where distance is in units of $1/\Gamma$. That this equation embodies Eq. (3) can be verified in any specific instance by writing out its matrix elements in a plane-wave representation. Equation (9) resembles a quantum mechanical Schrödinger equation with a time-dependent Hamiltonian, except here the dynamical evolution is in space rather than time. Indeed, we will refer to H as a Hamiltonian and to Eq. (9) as the Schrödinger picture [13] of the evolution of the state vector representing a temporal component of the electromagnetic field. We also think of Eq. (9) as expressing the flow of information through the medium.

The black-box view of the photorefractive medium is formally cast into place by taking the Heisenberg picture [13] of the evolution equations. We define the transformation, or two-beam coupling, operator T by

$$|\psi_\omega(z)\rangle = T(z)|\psi_\omega(0)\rangle . \tag{10}$$

From Eq. (9), the evolution of T is evidently given in this Heisenberg picture by

$$T'(z) = -iH_\rho(z)T(z) . \tag{11}$$

T is a unitary operator, $T^{-1} = T^\dagger$, reflecting the fact that the total intensity is preserved by the two-beam coupling interaction. From the definition of the density operator ρ and of the transformation operator T, $\rho(z) = T(z)\rho(0)T^\dagger(z)$, thus the evolution of the density operator is given by

$$\rho' = T'\rho(0)T^\dagger + T_0(0)T^{\dagger\prime} = -i[H_\rho, \rho] , \tag{12}$$

or

$$\rho' = [[\sigma_0, \rho], \rho]/4 . \tag{13}$$

Equations (11) and (13) are the primary results of this paper. They express the flow of information through the medium in different forms. T is the identity operator at $z = 0$, so knowing $T(L)$ and the fields at the input provides the fields at the output of the medium. If the field suddenly changes, the new field undergoes the old transformation until the medium has had time to respond, typically a fraction of a second. On the other hand, the density operator determines the specific behavior of an interaction and tracks the correlation among the modes.

The simplest case of beam coupling between two plane waves illustrates the operator formalism. The density and coupling matrices are

$$\rho = \frac{1}{I}\begin{pmatrix} I_+ & e_+e_-^* \\ e_+^*e_- & I_- \end{pmatrix}, \tag{14}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Hence the Hamiltonian is

$$H_\rho = \frac{i}{2I}\begin{pmatrix} 0 & e_+e_-^* \\ -e_+^*e_- & 0 \end{pmatrix}, \tag{16}$$

and the double-bracket equation gives

$$\begin{pmatrix} I'_+ & (e_+e_-^*)' \\ (e_+^*e_-)' & I'_- \end{pmatrix} = \frac{1}{I}\begin{pmatrix} I_+I_- & (I_- - I_+)e_+e_-^*/2 \\ (I_- - I_+)e_+^*e_-/2 & -I_+I_- \end{pmatrix}. \tag{17}$$

The diagonal elements of this equation are identical with Eq. (2) (other than the implicit factor of Γ), as they should be.

The two-beam coupling operator equations can be integrated directly when the Hamiltonian is spaitally commuting, that is $[H_\rho(z_1), H_\rho(z_2)] = 0$ for all z_1, z_2. In particular, the commuting assumption about the Hamiltonian allows us to formally integrate Eq. (11),

$$T(z) = \exp\left[-i\int_0^z H_\rho(\tilde z)\, d\tilde z\right]T(0) . \tag{18}$$

The expression for T can be put into closed form. To do so it is helpful to first cast the two-beam coupling equations in a "coupling" picture that treats the coupling operator as the dynamical variable while the fields remain fixed. We introduce

$$\sigma(z) \equiv T^\dagger\sigma_0 T . \tag{19}$$

The utility of this picture rests in the fact that we can solve for $\sigma(z)$ in all cases (whether or not the Hamiltonian commutes with itself at different positions), and the fact the equation of motion for the two-beam coupling operator can be cast entirely in terms of σ and its derivatives. Taking the spatial derivative we find that

$$\sigma' = [\sigma, [\sigma, \rho_0]]/4 = (\rho_0 - \sigma\rho_0\sigma)/2 , \tag{20}$$

where $\rho_0 \equiv \rho(0)$ and we have made use of the fact that $\sigma^2 = 1$. The solution to this equation is given by an explicit similarity transformation,

$$\sigma(z) = S\sigma_0 S^{-1}, \tag{21}$$

with the transformation operator given by

$$S(z) = \cosh(\rho_0 z/2) + \sinh(\rho_0 z/2)\sigma_0 . \tag{22}$$

That Eq. (21) satisfies (20) can be quickly verified by noting that $S' = \rho_0 S\sigma_0/2$ and by using the derivative of an inverse operator $(S^{-1})' = -S^{-1}S'S^{-1}$. Multiplication of the right-hand side of Eq. (11) on the left by $TT^\dagger = 1$ reexpresses the evolution of the two-beam coupling operator,

$$T' = -iTH_\sigma = T\sigma\sigma'/2 , \tag{23}$$

where

$$H_\sigma \equiv i[\sigma, \rho_0]/4 = \sigma\sigma'/2 = T^\dagger H_\rho T . \tag{24}$$

Now we are in a position to prove that the solution of the two-beam coupling operator for the spatially commuting Hamiltonian case is

$$T(z) = \sqrt{\sigma_0\sigma(z)} . \tag{25}$$

First, we see that the above satisfies the boundary condition, $T(0) = 1$. Second, we take $(T^2)' = T'T + TT' = 2TT' = T^2\sigma\sigma' = \sigma_0\sigma'$, which is consistent with the assumed form of the operator. Note that the second equality is a consequence of Eq. (18) and the boundary condition on T.

One can show that the Hamiltonian is spatially commuting for the "pure case," a condition defined by $\rho_0^2 = \rho_0$, which physically is the simplest case of two interacting fully mutual coherent temporal modes, such as those of Eq. (1). Note that it does not necessarily mean two plane waves; the modes may be composed of an arbitrary number of plane-wave components. One can also show that H is spatially commuting when $[\rho_0, \sigma_0\rho_0\sigma_0] = 0$.

This latter condition is satisfied by *any* 2×2 density matrix, therefore any two-mode problem including two modes that are partially mutually coherent in time. The latter condition is also satisfied by the general case of reflexive two-beam coupling in photorefractive media, a special instance of many interacting modes that is of particular interest in information processing [14].

What general statements can we now make about the connection between two-beam coupling and information processing? Despite nonlinear interaction, the resultant transformation remains unitary. Consequently both the trace and the eigenvalues of the density matrix are constants of the evolution. Equation (13) is, in fact, a special case of double-bracket equations, which have been studied extensively in the context of information processing for the class of symmetric matrices [15–17]. It is known, for example, that Eq. (13) describes a gradient ascent to the maximum of $\mathrm{Tr}(\rho\sigma_0)$ (Heisenberg picture) or of $\mathrm{Tr}(\rho_0\sigma)$ (coupling picture). Steady state corresponds to the condition that ρ commutes with σ_0. Therefore the density operator in the experimenter's representation asymptotically approaches a block diagonal.

The density matrix is composed of a weighted sum of state-vector outer products. The weights take on statistical meaning if one interprets the ratios for the various temporal components ω as probabilities, $P_\omega = I_\omega/I$. The interpretation is particularly appropriate for problems such as turbulence mitigation for which the statistical properties, but not the details of the optical field, can be characterized. The diagonal elements of the density matrix can also be interpreted as probabilities, and they are the natural dynamical variables to follow in a given two-beam coupling problem. Reflecting the gradient ascent character of the dynamics, the diagonal elements corresponding to the plus port monotonically increase towards their asymptotic values, while those corresponding to the minus port monotonically decrease towards their asymptotic values. By contrast, the intensities associated with individual state vectors do not necessarily evolve monotonically. Numerical calculations reveal, in fact, that energy can flow from the plus to the minus port, the reverse of events in simple two-beam coupling.

The ability to draw general conclusions about the dynamics owes much to the representation free form of operator methods. For a given numerical calculation one can choose a representation that is convenient for the problem at hand. Equations (13) and/or (16) are well behaved and easily evaluated numerically by standard integration packages whether or not H is spatially commuting. The various conserved quantities are all contained in the formalism and one need only specify the boundary condition. Once obtained, the black-box operator T provides easy access to other quantities of interest.

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