Parametric Amplification of Gravitational Fluctuations during Reheating

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Cosmological perturbations can undergo amplification by parametric resonance during preheating even on scales larger than the Hubble radius, without violating causality. A unified description of gravitational and matter fluctuations is crucial to determine the strength of the instability. To extract specific signatures of the oscillating inflaton field during reheating, it is essential to focus on a variable describing metric fluctuations which is constant in the standard analyses of inflation. For a massive inflaton without self-coupling, we find no additional growth of superhorizon modes during reheating beyond the usual predictions. For a massless self-coupled inflaton, there is a sub-Hubble scale resonance. [S0031-9007(99)08452-5]

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As was initially realized in [1] and worked out in detail in [2–4] and many other papers [5], parametric resonance instabilities play a crucial role in the dynamics of the reheating of an inflationary Universe. Reheating (or now more accurately called "preheating" [2]) is the period after inflation when the inflaton, the scalar field driving the period of exponential expansion, oscillates coherently about its ground state and gradually transforms its energy into the matter and radiation of which the present Universe is made up.

The parametric resonance instability can be seen by considering the linearized equation of motion of fields χ which couple to the inflaton. Neglecting for a moment the expansion of the Universe, the equation of motion for the Fourier modes χ_k becomes a harmonic oscillator equation with a periodically varying mass, the Mathieu equation. It is well known that this equation admits instability bands, regions of k for which the solutions grow exponentially. The instabilities persist after taking the expansion of the Universe into account.

The parametric resonance instabilities have important consequences for cosmology. They will lead to a reheating temperature which can be much larger than would be obtained by calculating the efficiency of reheating using perturbation theory. This could have important implications for grand-unification-scale baryogenesis [6], the formation of topological defects [7], or the production of supermassive dark matter [8].

Initially [1], the parametric instability was discussed in a model in which the resonance bands are narrow. It was then pointed out [2] that, in models of chaotic inflation, the instability bands can be much broader, and the Floquet exponent μ_k giving the rate of exponential growth correspondingly larger. In [9] it was discovered that a negative sign of the coupling constant between the inflaton and the χ field leads to an enhanced instability ("negative coupling instability"). White noise eliminates the stability bands all together [10,11].

The inflaton field couples to linearized metric perturbations, and, hence, gravitational waves may also experience parametric resonant amplification [12] during reheating. The growth of scalar metric perturbations due to the oscillations of the inflaton field was first considered by Nambu and Taruya and by Kodama and Hamazaki [13]. Nambu and Taruya concluded that scalar perturbations are amplified during reheating, but did not compare their growth with the usual growth of cosmological perturbations. Kodama and Hamazaki focused on the evolution of the "Bardeen parameter," a gauge invariant measure of the cosmological perturbations which in the usual analysis of the growth of fluctuations (which neglects the oscillations of the inflaton field) is constant in time for modes larger than the Hubble radius. They concluded that, in spite of singular terms in the perturbation equations, the Bardeen parameter remains constant. Bassett, Kaiser, and Maartens [14] have reanalyzed this problem and argue that there is a negative coupling parametric resonance instability which leads to a rapid growth of metric perturbations.

Here, we analyze the growth of metric inhomogeneities during reheating in a more complete way, making use of the gauge-invariant theory of perturbations (see [15] for a review). Since matter and metric are coupled by the Einstein constraint equations, the fluctuations can be described completely by a single gauge-invariant variable Φ. In longitudinal gauge, the perturbed metric can be written in terms of Φ as

$$ds^2 = dt^2(1 + 2\Phi) - a^2(t)(1 - 2\Phi), \qquad (1)$$

where $a(t)$ is the scale factor. As pointed out in [13], two of the coefficients in the equation of motion for Φ are singular due to the oscillations of the inflaton field. As realized in [13], the divergence disappears if, instead of Φ, one considers the equation of motion for the Sasaki-Mukhanov variable Q [16], a variable in terms of which the quantization of cosmological perturbations is straightforward (see [15] for a review). We demonstrate that an instability persists in the equation

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of motion for a rescaled variable $\tilde{Q}$. This instability, however, is not of negative coupling type. For a massive inflaton, it leads only to an increase of $\tilde{Q}$ proportional to $a^{3/2}(t)$ for long wavelength fluctuations. Hence, the amplitude of Q is constant in time, and there is no amplification of fluctuations beyond what the usual theory predicts. However, for a massless self-coupled inflaton. $\tilde{Q}$ experiences an initial increase. We show why the usual methods to study the evolution of perturbations in inflationary cosmology miss the possible additional growth of fluctuations due to the oscillating inflaton field.

Our starting point is the equations of motion for the perturbations of the Einstein-Higgs system about a Friedmann-Robertson-Walker background solution. In terms of the gauge invariant metric and matter variables Φ [see (1)] and $\delta\phi_{\mathrm{gi}}$ (which is longitudinal gauge is equal to the scalar field perturbation $\delta\phi$), the system of equations in momentum space is

$$\ddot{\Phi} + 3H\dot{\Phi} + \left[\frac{k^2}{a^2} + 2(\dot{H} + H^2)\right]\Phi = \kappa^2(\ddot{\phi} + H\dot{\phi})\delta\phi\,, \qquad (2)$$

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(\frac{k^2}{a^2} + V''\right)\delta\phi = 4\dot{\Phi}\dot{\phi} - 2V'\Phi\,, \qquad (3)$$

$$\dot{\Phi} + H\Phi = \frac{1}{2}\kappa^2\dot{\phi}\,\delta\phi\,, \qquad (4)$$

where $\kappa^2 = 8\pi G$, $H = \dot{a}/a$ is the Hubble expansion rate, ϕ is the homogeneous background field for a scalar matter field with potential energy density $V(\phi)$, and a prime denotes the derivative with respect to ϕ. Equation (3) is the equation of motion for $\delta\phi$, (4) is the Einstein momentum constraint equation, and (2) is a combination of the dynamical equation of motion for Φ and the Einstein energy constraint equation [see Eqs. (6.42) and (6.40) of [15]]. If physical time is replaced by conformal time, these equations yield Eqs. (2)–(4) of [14].

Because of (4), there is only one physical degree of freedom, chosen to be Φ. Note that the "source" term in (2) is crucial [see (4)]. The correct equation of motion for Φ is obtained by inserting (4) into (2), with the result

$$\ddot{\Phi} + \left(H - 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\Phi} + \left(\frac{k^2}{a^2} + 2\dot{H} - 2H\frac{\ddot{\phi}}{\dot{\phi}}\right)\Phi = 0\,. \qquad (5)$$

During the slow rolling period of an inflationary cosmology, the coefficients in this equation are well behaved. However, oscillations of ϕ during reheating lead to singularities. These singularities can be eliminated [13] by making use of the Sasaki-Mukhanov variable [16] Q, the combination

$$Q = \delta\phi + \frac{\dot{\phi}}{H}\Phi \qquad (6)$$

of the gauge-invariant matter and metric perturbations in terms of which the unified quantization of the matter and metric perturbations is easy. In terms of Q, the equation of motion (5) becomes [13,15]

$$\ddot{Q} + 3H\dot{Q} + \left[V'' + \frac{k^2}{a^2} + 2\left(\frac{\dot{H}}{H} + 3H\right)^{\cdot}\right]Q = 0\,. \qquad (7)$$

As is evident, the coefficients of this differential equation are regular. Given Q, it is possible to obtain Φ since (6) can be rewritten in the form

$$\frac{k^2}{a^2}\Phi = \frac{\kappa^2}{2}\frac{\dot{\phi}^2}{H}\left(\frac{H}{\dot{\phi}}Q\right)^{\cdot}. \qquad (8)$$

The Hubble damping term in Eq. (7) for Q can be eliminated by introducing the rescaled variable $\tilde{Q}$. For a massive inflaton with potential $V(\phi) = m^2\phi^2/2$, $\tilde{Q} = a^{3/2}Q$. In terms of $\tilde{Q}$, (7) becomes

$$\ddot{\tilde{Q}} + \left[V'' + \frac{k^2}{a^2} + 2\left(\frac{\dot{H}}{H} + 3H\right) - \frac{9}{4}\left(H^2 + \frac{2}{3}\dot{H}\right)\right]\tilde{Q} = 0\,. \qquad (9)$$

Making use of the background Einstein equations, (9) can be written as

$$\ddot{\tilde{Q}} + \left[V'' + \frac{k^2}{a^2} + 3\kappa^2\dot{\phi}^2 + 2\kappa^2\frac{\dot{\phi}V'}{H} - \frac{\kappa^4}{2H^2}\dot{\phi}^4 + \frac{3\kappa^2}{4}p_\phi\right]\tilde{Q} = 0\,, \qquad (10)$$

where p_ϕ is the background pressure of the scalar field.

After the period of slow rolling has ended, the value of H is smaller than m. Hence, it follows from the background equation of motion for ϕ that—in the absence of back reaction and with accuracy increasing in time—the motion of $\tilde{\phi} = a^{3/2}\phi$ is oscillatory in time. In this limit, Eq. (9) has the form

$$\ddot{\tilde{Q}} + [A(k) - 2q\cos(mt)]\tilde{Q} = 0\,, \qquad (11)$$

where

$$A(k) = m^2 + \frac{k^2}{a^2} + r\,, \qquad (12)$$

where r contains the time average of the last four terms in the square bracket of (10), and q contains the coefficients of the oscillating parts of these terms. Since $q(t)$ is decreasing, (11) is not of the form of the usual parametric resonance equation, and no exponentially growing solutions will result.

The second of the four last terms in the square bracket of (10) is the most important. Its initial amplitude is the largest, and it decays the least fast as a function of time. Approximating $a(t) \sim t^{2/3}$ corresponding to a pressureless phase, it is easy to check that the second term decays as t^{-1}, whereas the other three terms decay as t^{-2}. Note that the decay rate of the dominant term of q as a function of time is less fast than the corresponding decay rate of q for matter fluctuations [4], a point already

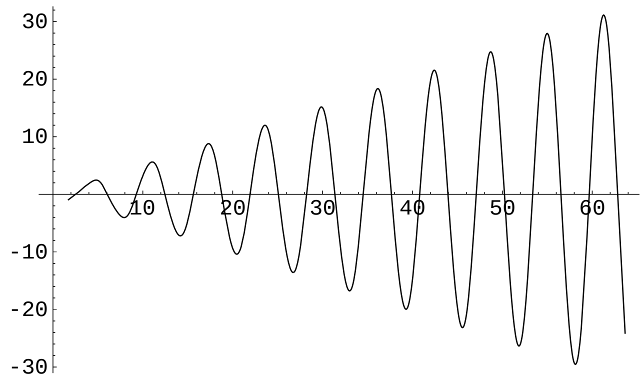

FIG. 1. Evolution of $\tilde{Q}$ as a function of time in the case of a massive inflaton for a mode with $k = 0.1$ m, with initial conditions such that $\tilde{Q}$ is -1 and its time derivative 0 at the initial time when $\phi = 0.2\ \mathrm{m}_{pl}$, after slow rolling has ended. Time is expressed in units of m^{-1}.

emphasized by [14]. The amplitude of this leading term in q starts out slightly larger than m^2. Because of the expansion of the Universe, the instability does not lead to exponential increase in $\tilde{Q}$, but only to an increase proportional to $a^{3/2}(t)$ (see Fig. 1), which implies that the amplitude of Q remains constant.

The results are different for a massless self-coupled inflaton with potential $V(\phi) = \lambda\phi^4/4$. In this case, we eliminate (following [17]) the Hubble damping term in (7) by introducing the rescaled variable $\tilde{Q} = a(\eta)Q$ and by working in terms of conformal time η. In this case, (7) becomes

$$\ddot{\tilde{Q}} + \left[A(k) + 3\,\mathrm{cn}^2\!\left(x, \frac{1}{\sqrt{2}}\right)\right]\tilde{Q} = 0\,, \tag{13}$$

where $x = \sqrt{\lambda}\,\mathcal{A}_\phi \eta$ is a rescaled conformal time, $\mathcal{A}_\phi$ is the amplitude of oscillation of $a(\eta)\phi$, cn stands for the elliptic cosine function, and

$$A(k) = \frac{k^2}{\lambda \mathcal{A}_\phi^2} + s\,. \tag{14}$$

Here, s stands for oscillatory terms which are important initially and lead to an increase in Q (even for $k^2 \ll \lambda\mathcal{A}_\phi^2$), but which decay in time as η^{-1}. For $s = 0$, Eq. (13) is a Lamé equation, has been studied in detail in [17], and exhibits no resonance for $k^2 \ll \lambda\mathcal{A}_\phi^2$.

We have solved the equation of motion (10) numerically. Figure 1 shows the resulting time evolution of the rescaled variable $\tilde{Q}$ in the case of a massive inflaton for the mode $k = 0.1$ m (about 30 times larger wavelength than the Hubble radius) over a period of several oscillations of the background field. As mentioned above, the amplitude of $\tilde{Q}$ grows linearly in time (and thus the amplitude of Q remains constant) to within the numerical accuracy.

According to the usual treatment (see, e.g., [15]), the details of the equation of state are irrelevant for the final amplitude of fluctuations on scales larger than the reheating Hubble radius. If oscillations in the equation of state would lead to an increase in the amplitude of Φ, this could have important implications for the spectrum of density and cosmic microwave background (CMB) fluctuations [14]. Such an effect would *not* violate causality. Inflation has already set up fluctuations on scales larger than the reheating Hubble radius, and the inflaton field is oscillating coherently on these scales, which can lead to a self-gravitational increase in the magnitude of fluctuations without violating causality, analogous to the usual evolution of quantum fluctuations in inflationary cosmology.

The clearest way to determine if the growth of $\tilde{Q}$ calculated above is a new effect is to calculate the quantity ζ:

$$\zeta = \frac{2}{3}\,\frac{\Phi + H^{-1}\dot{\Phi}}{1 + w} + \Phi\,, \tag{15}$$

which according to the standard cosmological perturbation theory remains constant for modes with wavelength larger than the Hubble radius [15]. Here, $w = p/\rho$ describes the equation of state, and p and ρ are the pressure and energy density, respectively. More precisely [15], $\dot{\phi}^2\dot{\zeta} = 0$ is equivalent to the equation of motion for Φ (for modes with wavelength larger than the Hubble radius). Hence, it is usually deduced that $\dot{\zeta} = 0$. This conclusion, however, may break down if $\dot{\phi} = 0$ which occurs during reheating. Hence, it is possible that additional resonant amplification of fluctuations during reheating occurs.

The variable ζ is related to Q via

$$\zeta = \frac{H}{\dot{\phi}}\,Q\,. \tag{16}$$

Since the amplitudes of H and $\dot{\phi}$ decrease at the same rate (proportional to t^{-1}) if we assume that $a(t) \sim t^{2/3}$, the constancy of the amplitude of Q (for a massive inflaton) leads to the conclusion that ζ is constant when evaluated at the same phase during each oscillation period of the inflaton ϕ. Since $1 + w$ has constant amplitude during the period of oscillation, we deduce from Fig. 2 that the amplitude of ζ is constant. Hence [inasfar as the $\dot{\Phi}$ term in (15) can be neglected], although Φ oscillates even on large scales, its amplitude remains constant. This can also be seen by evaluating Φ directly using (8).

Note that Φ is the basic physical quantity which is well-defined at all times. It is the quantity which determines the power spectrum of density fluctuations and of CMB anisotropies. In contrast, ζ is an auxiliary quantity. At each zero of $\dot{\phi}$ there is a singularity in the relation between Φ and ζ. What is therefore important is to calculate the value of ζ for each zero crossing of ϕ and compare the values. As seen from Fig. 2 [plot of $(1 + w)\zeta$ as a function of time, determined directly from (16)], ζ does not change over a period. This demonstrates that the net growth of $\tilde{Q}$ observed in Fig. 1 agrees with the results of the usual analysis of the growth of cosmological perturbations.

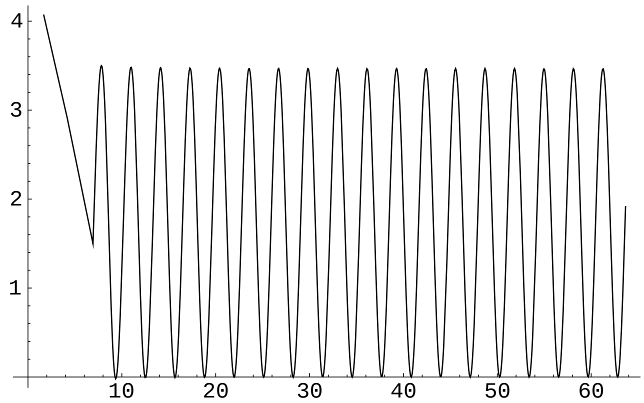

FIG. 2. Evolution of $(1 + w)\zeta$ for a theory with a massive inflaton, and for same mode, initial conditions, and initial time as in Fig. 1.

In summary, we have studied the growth of cosmological perturbations during a phase of coherent oscillations of the scalar field driving inflation, which leads to singularities in $c_s^2 = \dot{p}/\dot{\rho}$. In this period, the equations of motion of the fluctuations are similar in structure to the equations describing coupled matter fields during reheating. As emphasized in [4,9,17], the presence or absence of a mass in the inflaton field is crucial. To study the evolution of cosmological perturbations, it is crucial to work in terms of variables in terms of which the singularities as absent, and in which the usual growth of cosmological fluctuations (obtained without taking into account the oscillations of ϕ) is factored out, such as ζ.

For a massive inflaton, we find no amplification of long wavelength fluctuations (wavelength larger than the Hubble radius during reheating)—the amplitude of ζ is constant. A modeling of the reheating period including the oscillations of ϕ will lead to the same growth as is obtained in the usual analyses of perturbations in which the transition from the inflationary phase to the postinflationary radiation-dominated period is modeled (implicitly) by a monotonous transition in the equation of state, or in which the background quantities are averaged in time [18]. In a comparison Letter [19], Parry and Easther demonstrate that even a full nonlinear analysis does not lead to any additional growth of fluctuations with wavelength larger than the Hubble radius during reheating.

However, for a massless self-coupled inflaton, there is evidence of a stable resonance band (see [20] for a detailed discussion, and [21] for related work on gravitational back reaction).

Note that the negative coupling instability discussed in [14] is not present in the relevant Eq. (7) for $\tilde{Q}$. The negative coupling instability in the equation for Φ is precisely the instability which is responsible for the amplification of the fluctuations in the standard analysis of the growth of cosmological perturbations.

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