Conformal "Thin-Sandwich" Data for the Initial-Value Problem of General Relativity

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The initial-value problem is posed by giving a conformal three-metric on each of two nearby spacelike hypersurfaces, the proper-time separation of the hypersurfaces up to a multiplier to be determined, and the mean (extrinsic) curvature of one slice. The resulting equations have the *same* elliptic form as in the one-hypersurface formulation. The metrical roots of this form are revealed by a conformal "thin sandwich" viewpoint coupled with the transformation properties of the lapse function. [S0031-9007(99)08400-8]

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In this paper I propose a new interpretation of the four Einstein vacuum initial-value constraints. (The presence of matter would add nothing new to the analysis.) Partly in the spirit of a "thin sandwich" viewpoint, I base this approach on prescribing the *conformal* metric [1] on each of two nearby spacelike hypersurfaces ("time slices" $t = t'$ and $t = t' + \delta t$) that make a thin sandwich (TS). Essential use is made of a new understanding of the role of the lapse function in general relativity [2,3]. The new formulation could prove useful both conceptually, and in practice, as a way to construct initial data in which one has a hold on the input data different from that in the currently accepted approach. The new approach allows us to *derive* from its dynamical and metrical foundations the important scaling law $\overline{A}^{ij} = \psi^{-10}A^{ij}$ for the traceless part of the extrinsic curvature. This rule is simply postulated in the one-hypersurface approach.

The new formulation differs from the well-known TS conjecture of Baierlein, Sharp, and Wheeler (BSW), in which the *full* spatial Riemannian metric $\overline{g}_{ij}$ is given on each of two infinitesimally separated hypersurfaces [4–6]. (The orthogonal separation $\overline{N}\delta t$ between the slices is assumed never to change signs in the BSW proposal and also here.) The four unknowns needed to solve the constraints were taken by BSW to be the "lapse function" $\overline{N}(x)$ and the spatial "shift vector" $\overline{\beta}^i(x)$ (see below). By using a known vacuum spacetime solution of Einstein's equations from which to obtain BSW data, one sees that their proposal must sometimes work. However, an analysis of the BSW proposal by Bartnik and Fodor [7] describes the general situation clearly, and one can only conclude that the BSW proposal is unsatisfactory. For example, an infinite number of nontrivial counterexamples to the BSW conjecture, based on compact three-geometries of negative scalar curvature with one (not ∞^1) constraint (fixed volume), have been described in [8].

The initial-value problem (IVP), that is, satisfying the four constraints, is fundamentally a *one*-hypersurface embedding problem. The four constraints are the Gauss-Codazzi embedding equations for a time slice in a Ricci-flat spacetime. They limit the allowed values of the metric $\overline{g}_{ij}$ and extrinsic curvature $\overline{K}_{ij}$ of an "initial" time slice in a yet-to-be constructed vacuum spacetime. This basic form will be referred to as the $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$ form, where Σ is the slice, say $t = t'$. In this case, the constraints have already been posed as a semilinear elliptic system for spatial scalar and spatial vector potentials, generalizations of the Newtonian potential [9–11]. A significant virtue of the formulation in this paper is that the constraints again become a semilinear elliptic system with the *same* essential mathematical structure as has the $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$ form. This surprising result, as we shall see, arises from the behavior of the lapse function [2,3].

The constraint equations are Σ are, in vacuum,

$$\overline{\nabla}_j(\overline{K}^{ij} - \overline{K}\,\overline{g}^{ij}) = 0\,, \tag{1}$$

$$R(\overline{g}) - \overline{K}_{ij}\overline{K}^{ij} + \overline{K}^2 = 0\,, \tag{2}$$

where $R(\overline{g})$ is the scalar curvature of $\overline{g}_{ij}$, $\overline{\nabla}_j$ is the Levi-Civita connection of $\overline{g}_{ij}$, and $\overline{K}$ is the trace of $\overline{K}_{ij}$, also called the "mean curvature" of the slice. (A review of this geometry is given in [11].) The overbar is used to denote quantities that satisfy the constraints.

The time derivative of the spatial metric $\overline{g}_{ij}$ is related to $\overline{K}_{ij}$, $\overline{N}$, and the shift vector $\overline{\beta}^i$ by

$$\partial_t\overline{g}_{ij} \equiv \dot{\overline{g}}_{ij} = -2\overline{N}\,\overline{K}_{ij} + (\overline{\nabla}_i\overline{\beta}_j + \overline{\nabla}_j\overline{\beta}_i)\,, \tag{3}$$

where $\overline{\beta}_j = \overline{g}_{ji}\overline{\beta}^i$. The fixed spatial coordinates $\vec{x}$ of a point on the "second" hypersurface, as evaluated on the "first" hypersurface, are displaced by $\overline{\beta}^i(\vec{x})\delta t$ with respect to those on the first hypersurface, with an orthogonal link from the first to the second surface as a fiducial reference: $\overline{\beta}_i = \frac{\partial}{\partial t} * \frac{\partial}{\partial x^i}$, where $*$ is the physical spacetime inner product of the indicated natural basis four-vectors. The essentially arbitrary direction of $\frac{\partial}{\partial t}$ is why $\overline{N}(x)$ and $\overline{\beta}^i(x)$ appear in the TS formulation. In contrast, the tensor $\overline{K}_{ij}$ is always determined by the behavior of the unit normal on one slice and therefore does not possess the kinematical freedom, i.e., the gauge variance, of $\frac{\partial}{\partial t}$. Therefore, $\overline{N}$ and $\overline{\beta}^i$ do not appear in the one-hypersurface IVP for $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$.

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Turning now to the conformal metrics in the IVP, we recall that two metrics g_{ij} and $\bar{g}_{ij}$ are conformally equivalent if and only if there is a scalar $\psi > 0$ such that $\bar{g}_{ij} = \psi^4 g_{ij}$. The conformally invariant representative of the entire conformal equivalence class, in three dimensions, is the weight $(-2/3)$ unit-determinant "conformal metric" $\hat{g}_{ij} = \bar{g}^{-1/3}\bar{g}_{ij} = g^{-1/3}g_{ij}$ with $\bar{g} = \det(\bar{g}_{ij})$ and $g = (\det g_{ij})$. Note particularly that for any small perturbation, $\bar{g}^{ij}\delta\hat{g}_{ij} = 0$. We will use the important relation

$$\bar{g}^{ij}\partial_t\hat{g}_{ij} = g^{ij}\partial_t\hat{g}_{ij} = \hat{g}^{ij}\partial_t\hat{g}_{ij} = 0\,. \tag{4}$$

In the following, rather than use the mathematical apparatus associated with conformally weighted objects such as $\hat{g}_{ij}$, we find it simpler to use ordinary scalars and tensors to the same effect. Thus, let the role of $\hat{g}_{ij}$ on the first surface be played by a given metric g_{ij} such that the physical metric that satisfies the constraints is $\bar{g}_{ij} = \psi^4 g_{ij}$ for some scalar $\psi > 0$. (This corresponds to "dressing" the initial unimodular conformal metric $\hat{g}_{ij}$ with the correct determinant factor $\bar{g}^{1/3} = \psi^4 g^{1/3}$. This process does not alter the conformal equivalence class of the metric.) The role of the conformal metric on the second surface is played by the metric $g'_{ij} = g_{ij} + u_{ij}\delta t$, where, in keeping with (4), the velocity tensor $u_{ij} = \dot{g}_{ij}$ is chosen such that

$$g^{ij}u_{ij} = g^{ij}\dot{g}_{ij} = 0\,. \tag{5}$$

Then, to first order in δt, g'_{ij} and g_{ij} have equal determinants, as desired; but g_{ij} and g'_{ij} are not in the same conformal equivalence class in general.

We now examine the relation between the covariant derivative operators ∇_i of g_{ij} and $\bar{\nabla}_i$ of $\bar{g}_{ij}$. The relation is determined by

$$\bar{\Gamma}^i_{jk}(\bar{g}) = \Gamma^i_{jk}(g) + 2\psi^{-1}(2\delta^i_{(j}\partial_{k)}\psi - g^{il}g_{jk}\partial_l\psi)\,, \tag{6}$$

from which follows the scalar curvature relation first used in an initial-value problem by Lichnerowicz [12],

$$R(\bar{g}) = \psi^{-4}R(g) - 8\psi^{-5}\Delta_g\psi\,, \tag{7}$$

where $\Delta_g\psi \equiv g^{kl}\nabla_k\nabla_l\psi$ is the "rough" scalar Laplacian associated with g_{ij}.

Next, we solve (3) for its traceless part

$$\dot{\bar{g}}_{ij} - \tfrac{1}{3}\bar{g}_{ij}\bar{g}^{kl}\dot{\bar{g}}_{kl} \equiv \bar{u}_{ij} = -2\bar{N}\bar{A}_{ij} + (\bar{L}\bar{\beta})_{ij} \tag{8}$$

with $\bar{A}_{ij} \equiv \bar{K}_{ij} - \frac{1}{3}\bar{K}\bar{g}_{ij}$ and

$$(\bar{L}\bar{\beta})_{ij} \equiv \bar{\nabla}_i\bar{\beta}_j + \bar{\nabla}_j\bar{\beta}_i - (2/3)\bar{g}_{ij}\bar{\nabla}^k\bar{\beta}_k\,. \tag{9}$$

Expression (9) vanishes, for nonvanishing $\bar{\beta}^i$, if and only if $\bar{g}_{ij}$ admits a conformal Killing vector $\bar{\beta}^i = k^i$. Clearly, k^i would also be a conformal Killing vector of g_{ij}, or of any metric conformally equivalent to $\bar{g}_{ij}$, with no scaling of k^i. This teaches us that in general $\bar{\beta}^i = \beta^i$, while $\bar{\beta}_i = \bar{g}_{ij}\bar{\beta}^j = \psi^4 g_{ij}\beta^j = \psi^4\beta_i$. That $\bar{\beta}^i = \beta^i$ also follows because β^i, generator of a spatial diffeomorphism, is not a dynamical variable. The latter "rule" was inferred to be a matter of principle.

It is clear in (8) that the left hand side $\bar{u}_{ij}$ satisfies $\bar{u}_{ij} = \psi^4 u_{ij}$ because the terms in $\dot{\psi}$ cancel out. Furthermore, a straightforward calculation shows that

$$(\bar{L}\bar{\beta})_{ij} = \psi^4(L\beta)_{ij}; \qquad (\bar{L}\beta)^{ij} = \psi^{-4}(L\beta)^{ij}. \tag{10}$$

Next, we note, perhaps surprisingly, that the lapse function $\bar{N}$ has essential nontrivial conformal behavior. Furthermore, this is *the* new element in the IVP analysis. In [2,3,13,14] the "slicing function" $\alpha(t,x) > 0$ replaces the lapse function $\bar{N}$,

$$\bar{N} = \bar{g}^{1/2}\alpha\,, \tag{11}$$

with important improvements then appearing in Teitelboim's path integral [13], in Ashtekar's new variables program [14], in the canonical action principle [2,3], and in making clear the role of the contracted Bianchi identities [2,3]. The lapse is now a dynamical variable because of the $\bar{g}^{1/2}$ factor [2,3,14]. Furthermore, in the construction of mathematically hyperbolic systems for the Einstein *evolution* equations with explicitly physical characteristics, and only such (for example, [15–17]), it turns out to be $\alpha(t,x)$, not the usual lapse function $\bar{N}$, that can be freely specified. This use of $\bar{N} = \bar{g}^{1/2}\alpha$ is Choquet-Bruhat's "algebraic gauge" [18,19] with, in general, a "gauge source" [20]. Actually, $\bar{N} = \bar{g}^{1/2}\alpha$ should be seen as a change of variables in which one specifies freely $\alpha(t,x) > 0$ rather than N. For these reasons, we conclude that α is not a dynamical variable, $\bar{\alpha} = \alpha$. For the lapse, we have from (11), with N given and positive,

$$\bar{N} = \psi^6 N\,. \tag{12}$$

Finally, we recall from the standard initial value problem for $(\Sigma, \bar{\mathbf{g}}, \bar{\mathbf{K}})$ that the separation of the extrinsic curvature into (its irreducible) trace and traceless parts is fundamental, as it is here, and that $\bar{K} = K$ [1]: the trace is not transformed even though it is apparently dynamical. It "anchors" the construction, setting a reference scale by fixing an observable dimensionful variable. (In closed worlds K is like a "time" variable, in that it may "locate" the thin sandwich. In cosmology, K is essentially the inverse mean "Hubble time.") There is no underlying geometrical derivation of $\bar{K} = K$, unlike the case of $\bar{A}_{ij}$ below. The conformal invariance of K is primitive. See the results in (19) below.

Now we solve (8) for $\bar{A}^{ij}$ and find

$$\begin{aligned}\bar{A}^{ij} &= \psi^{-6}(2N)^{-1}[\psi^{-4}(L\bar{\beta})^{ij} - \psi^{-4}u^{ij}] \\ &= \psi^{-10}\{(2N)^{-1}[(L\bar{\beta})^{ij} - u^{ij}]\} = \psi^{-10}A^{ij},\end{aligned} \tag{13}$$

the same conformal scaling that was postulated by Lichnerowicz [12] and others [9–11] for the traceless part of $\bar{K}^{ij}$ in the one-hypersurface problem. One now has a derivation of this fundamental transformation from its metrical foundations. The momentum constraint (1) becomes

$$\nabla_j[(2N)^{-1}(L\bar{\beta})^{ij}] = \nabla_j[(2N)^{-1}u^{ij}] + (2/3)\psi^6\nabla^i K\,, \tag{14}$$

for unknown $\overline{\beta}^i$ and known N, g_{ij}, u_{ij}, and K. The operator on the left, being in elliptic "divergence form" with $N > 0$, does not differ in any important property from its counterpart in the $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$ analysis [9–11]. The Hamiltonian constraint (2) becomes [21]

$$8\Delta_g \psi - R(g)\psi + A_{ij}A^{ij}\psi^{-7} - (2/3)K\psi^5 = 0, \quad (15)$$

for unknown ψ, where A^{ij} is given in (13). This equation has precisely the same form in the one-hypersurface and two-hypersurfaces constraint problems. Note that (14) and (15) are not coupled if $K = \text{const}$; i.e., one solves (14), then (15). *No tensor splittings* [21,22] are needed in the new formulation of the constraints.

Thus, the free data are $\{g_{ij}, u_{ij}, N, K\}$ and the solution is $\{\psi, \overline{\beta}^i\}$. Mathematical analysis of the corresponding elliptic system (14, 15) has been carried out elsewhere, for example, [9,10,23–25], and will not be repeated here. The corresponding situation in the $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$ analysis is that the free data are $\{g_{ij}, A_{ij}, K\}$ and the solution is $\{\varphi, W^i\}$, where W^i is obtained from a tensor splitting of A_{ij} [21,22]. Note that $\varphi \neq \psi$ and $W^i \neq \overline{\beta}^i$. Only part of A_{ij}, found in the splitting, is free. The conformal covariance of the new method, i.e., starting with different representatives of a given conformal equivalence class is *unique* and clear. On the other hand, that of the $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$ analysis can follow two inequivalent routes because there are two slightly different conformal analyses possible for construction of $(\Sigma, \overline{\mathbf{g}}, \overline{\mathbf{K}})$. This nonuniqueness arises because conformal scaling and tensor splittings are not commutative in a straightforward way. The method of tensor splitting in [11] gives the Hamiltonian constraint in the form of (15).

These data are not in perfect analogy to those conjectured by BSW, because K and N can be thought of as belonging to the thin sandwich as a whole. The role of K has been described. The role of $N = g^{1/2}\alpha$ is to give the thickness of the sandwich, $\overline{N}\delta t$, in proper time measured orthogonally from $t = t'$ to $t = t''$:

$$\overline{N}\delta t = (\overline{g}^{1/2}\alpha)\delta t = (\psi^6 g^{1/2})\alpha\,\delta t = \psi^6(N\delta t). \quad (16)$$

The final relationships between the two physical Riemannian metrics $\overline{g}_{ij}$ and $\overline{g}'_{ij} = \overline{g}_{ij} + \dot{\overline{g}}_{ij}\delta t$ and the given data g_{ij} and $g'_{ij} = g_{ij} + u_{ij}\delta t$ are quite interesting. Of course, $\overline{g}_{ij} = \psi^4 g_{ij}$ is clear. But we have to calculate the relationship between $\overline{g}_{ij}$ and $\overline{g}'_{ij}$ as $\overline{g}'_{ij} = \overline{g}_{ij} + \dot{\overline{g}}_{ij}\delta t$, where, as in (3),

$$\begin{aligned}\dot{\overline{g}}_{ij} &= \partial_t(\psi^4 g_{ij}) \\ &= -2\overline{N}(\overline{A}_{ij} + \tfrac{1}{3}\overline{g}_{ij}K) + (\overline{\nabla}_i\overline{\beta}_j + \overline{\nabla}_j\overline{\beta}_i). \quad (17)\end{aligned}$$

Working out (17) gives a key result, namely,

$$\begin{aligned}\dot{\overline{g}}_{ij} &= \psi^4[u_{ij} + g_{ij}\partial_t(4\log\psi)] \\ &= \overline{u}_{ij} + \overline{g}_{ij}\partial_t(4\log\psi), \quad (18)\end{aligned}$$

where

$$\begin{aligned}\partial_t(4\log\psi) &= \tfrac{2}{3}(\nabla_k\overline{\beta}^k + 6\overline{\beta}^k\partial_k\log\psi - NK\psi^6) \\ &= \partial_t(\overline{g}/g)^{1/3} = \tfrac{2}{3}(\overline{\nabla}_k\overline{\beta}^k - \overline{N}\,\overline{K}). \quad (19)\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{\overline{g}}_{ij} &= \psi^4[u_{ij} + \tfrac{2}{3}g_{ij}(\nabla_k\overline{\beta}^k + 6\overline{\beta}^k\partial_k\log\psi - NK\psi^6)] \\ &= \overline{u}_{ij} + \tfrac{1}{3}\overline{g}_{ij}(2\overline{\nabla}_k\overline{\beta}^k - 2\overline{N}K). \quad (20)\end{aligned}$$

We see that $\dot{\psi}$ and $\dot{\overline{g}}_{ij}$ are fully determined by the constraints and, in the last equality of (20), that the conformal invariance of β^k $(= \overline{\beta}^k)$ and K $(= \overline{K})$ are fully consistent, having led to the precise geometrically correct form of $\dot{\overline{g}}_{ij}$ by virtue also of $\overline{N} = \psi^6 N$.

This interpretation of the semilinear elliptic constraint system has interesting differences from earlier ones because the data and solutions are related more simply to the spacetime metric, though not in the manner that would be implied by ordinary conformal transformations of the spacetime metric. In this "conformal" TS form one can see explicitly the role of every part of the metric. The new formulation shows that the one-hypersurface and two-hypersurfaces initial-value problems are both viable once the full implications in general relatively of the "dynamical conformal structures" are understood. The two viewpoints can be thought of as roughly analogous to a Hamiltonian and to a Lagrangian view of the constraints; the former because using $\overline{K}_{ij}$ directly [9–11] is equivalent to using the initial canonical momentum $\overline{\pi}^{ij} = \overline{g}^{1/2}(\overline{K}\overline{g}^{ij} - \overline{K}^{ij})$, and the latter because $\dot{\overline{g}}_{ij}$ is the initial velocity. This striking correspondence hangs on the subtle role of the lapse function through the Choquet-Bruhat relation $\overline{N} = \overline{g}^{1/2}\alpha$ and on the corresponding conformal invariance of K postulated by the author [21] in going beyond Lichnerowicz's choice $K = 0$. The "conformal thin sandwich" aspect of the results reflects Wheeler's approach.

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