Solitary Waves for N Coupled Nonlinear Schrödinger Equations

F. T. Hioe
Department of Physics, St. John Fisher College, Rochester, New York 14618
(Received 25 August 1998)

A hierarchy of exact analytic solitary-wave solutions for N coupled nonlinear Schrödinger equations for which the nonlinear coupling parameters can change continuously and cover many regions is presented. Besides their potentially many practical applications to optical communication and multispecies Bose-Einstein condensates for couplings outside the special integrable cases, these analytically solvable cases for special initial conditions supplement and provide important links to and among the integrable cases. [S0031-9007(98)08318-5]

PACS numbers: 42.65.Tg, 05.45.Yv, 42.81.Dp

Nonlinearly coupled dynamical systems with two or more degrees of freedom have been subjects of considerable interest for many years [1]. These systems can exhibit many interesting features, including chaos. In rare cases that involve specific interaction parameters, a nonlinearly coupled dynamical system may be found to be integrable. For most interaction parameters, however, a nonlinearly coupled system is usually not integrable. We should note that "integrable" usually is taken to mean for all initial conditions. The analytic results we present in this Letter show that the coupled nonlinear system for which these results apply is analytically solvable for a wide range of specific initial conditions, even though the system may not be integrable. We should also note, however, that in return for restricting the initial condition the analytic results are applicable for a wide range of nonlinear coupling parameters, and provide not only many more useful physical applications but also important links to the integrable cases. The coupled dynamical system we discuss in this Letter is closely related to the coupled nonlinear Schrödinger (CNLS) equations which have applications in many physical problems, especially in nonlinear optics and in the dynamics of Bose-Einstein condensates.

When two optical waves of different frequencies copropagate in a medium and interact nonlinearly through the medium, or when two polarization components of a wave interact nonlinearly at some central frequency, the propagation equations for the two problems can be considered together by considering the following N coupled nonlinear Schrödinger-like equations [2] for the case $N = 2$:

$$i\phi_{mz} + \phi_{mtt} + \kappa_m\phi_m + \left(\sum_{n=1}^{N} p_{mn}|\phi_n|^2\right)\phi_m + \left(\sum_{n=1}^{N} q_{mn}\phi_n^2\right)\phi_m^* = 0, \qquad m = 1,\ldots,N, \tag{1}$$

where $\phi_m(z,t)$ denotes the complex amplitude of the mth electric field envelope, or the mth polarization component, p's, q's, and κ's are parameters characteristic of the medium and interaction, and the subscripts in z and t denote derivatives with respect to z and t as opposed to the subscript m for different components. Equivalently, we may consider the following coupled equations:

$$i\psi_{mz} + \psi_{mtt} + \left(\sum_{n=1}^{N} p_{mn}|\psi_n|^2\right)\psi_m + \left(\sum_{n=1}^{N} q_{mn}\psi_n^2 e^{2i\kappa_n z}\right)\psi_m^* e^{-2i\kappa_m z} = 0, \qquad m = 1,\ldots,N, \tag{2}$$

which can be transformed into (1) with the substitutions $\psi_m = \phi_m \exp(-i\kappa_m z)$. Although the results presented in this Letter are for the specific case of $N = 2$, the method and prescription we present are, as will be seen, extendable to a general value of N for Eqs. (1) and (2).

We first search for the stationary-wave solution of the form

$$\phi_m(z,t) = x_m(t)\exp(i\Omega z), \tag{3}$$

where Ω is a real constant, and $x_m(t)$ are real functions of t only. Equations (1) reduce to the following, which we call the associated dynamical coupled nonlinear Schrödinger equations:

$$\ddot{x}_m - A_m x_m + \left(\sum_{n=1}^{N} b_{mn}x_n^2\right)x_m = 0, \qquad m = 1,\ldots,N, \tag{4}$$

where $\dot{x}$ denotes dx/dt, and where

$$A_m = \Omega - \kappa_{m'} \quad \text{and} \quad b_{mn} = p_{mn} + q_{mn}. \tag{5}$$

To eliminate the permutation symmetry, we arrange Eqs. (4) such that $A_1 \le A_2 \le \cdots \le A_N$. Since Eqs. (1) and (2) are invariant under a Galilean transformation,

 0031-9007/99/82(6)/1152(4)$15.00

traveling waves can be constructed from (3) by replacing $\phi_m(z,t)$ by

$$\phi_m(z, t - z/v) \exp\{i[t - z/2v]/2v\}, \qquad (6)$$

where v is the velocity of the waves.

We can identify negative (positive) values of b_{jk}, $k = 1,2$ with the normal (anomalous) group-velocity dispersion (GVD) region for ϕ_j. The special case of $A_j = 0$ and $b_{jk} = 1$ for $j,k = 1,2$ is associated with the known integrable case of Eqs. (1) first given by Manakov [3]. Various solitary-wave solutions for this case, which consist of the so-called bright and dark solitary waves, periodic (elliptic) waves, and waves of different forms, have been presented [4]. Other values of b's for which the coupled equations are integrable have been given [5]. The coupled equations (4) have been of interest and studied in nonlinear dynamics for many years, and they are known to be integrable for a number of specific values of A's and b's [1].

Let us refer to the space spanned by the N^2 real values of b_{jk}, $j,k = 1,\dots,N$ as the b space. Instead of asking whether, for some particular point of this b space, Eqs. (4) are integrable, the key idea behind the results presented in this Letter is to ask whether it is possible to postulate N analytic solutions for $x_1,\dots,x_N$, with variable parameters, and find regions in the b space for these solutions to hold so that, for these points or regions, Eqs. (4) have these analytic solutions, even though only for the initial conditions given by the values of these x's and $\dot{x}$'s at some initial time t_0. In this Letter, we show that there are many regions in the b space where the values for b's can change continuously over wide ranges and for which the coupled equations are analytically solvable. Specifically, we present a prescription for obtaining such regions and present sixteen analytically solvable regions for the case $N = 2$. The ansatz we use is that $x_1(t),\dots,x_N(t)$ be expressed in terms of N of the $2n + 1$ Lamé functions of order n [6], with repetition allowed (i.e., the same function for different x's) for $n = 1,\dots,N - 1$, and without repetition for $n = N$.

Let $h_j^{(n)}$, $j = 1,\dots,2n + 1$, arranged in descending order of magnitude, denote the characteristic values, and $f_j^{(n)}(u)$ the corresponding characteristic function (Lamé function), of the Lamé equation of order n, $d^2y/du^2 + [h - n(n + 1)k^2\mathrm{sn}^2(u,k)]y = 0$. We make the ansatz that

$$x_1(t) = \sqrt{C_1}\, f_p^{(n)}(\alpha t), \qquad x_2(t) = \sqrt{C_2}\, f_q^{(n)}(\alpha t), \quad (7)$$

be a solution of Eqs. (4) for $N = 2$, where $n = 1,2$, $p,q = 1,\dots,2n + 1$, $p \le q$ for $n = 1$, and $p < q$ for $n = 2$. Since $x_1(t)$ and $x_2(t)$ are assumed real, we require that C_1 and C_2 be real and positive. Substitutions of the ansatz (7) into Eq. (4) result in algebraic equations for the b's, A's, C's, and α and k^2 which can be expressed in a compact way in terms of three matrices $\mathbf{\Gamma}$, $\mathbf{B}$, and $\mathbf{D}$ which we define in the following. We start by expressing the square of the jth Lamé function of order n in a power series in $s = \mathrm{sn}(u,k)$ as

$$[f_j^{(n)}(u)]^2 = \sum_{i=1}^{n+1} a_{ij}^{(n)} s^{2(i-1)}, \qquad j = 1,\dots,2n + 1. \qquad (8)$$

We form a $(n + 1) \times (2n + 1)$ matrix $\mathbf{a} = [a_{ij}^{(n)}]$, i.e., a 2×3 matrix for $n = 1$ and a 3×5 matrix for $n = 2$. We define $\mathbf{\Gamma} = [c_{ij}]$ to be a $(n + 1) \times 2$ matrix, where $c_{i1} = a_{ip}^{(n)} C_1$, $c_{i2} = a_{iq}^{(n)} C_2$, $i = 1,\dots,n + 1$, where C's are the amplitudes in (7). $\mathbf{B} = [b_{ij}]$, $i,j = 1,2$, is a 2×2 matrix, where b's are the nonlinear coupling parameters in (4). $\mathbf{D} = [d_{ij}^{(n)}]$, $i = 1,\dots,n + 1$, $j = 1,2$, is a $(n + 1) \times 2$ matrix, where $d_{11}^{(n)} = A_1 + h_p^{(n)}\alpha^2$, $d_{12}^{(n)} = A_2 + h_q^{(n)}\alpha^2$, $d_{2j}^{(n)} = -n(n + 1)k^2\alpha^2$, $d_{3j}^{(n)} = 0$, $j = 1,2$, A's are the linear coupling parameters in (4), and $h_j^{(n)}$'s are the characteristic values of the Lamé equation. The algebraic equations that need to be satisfied for (7) to be a solution of Eqs. (4) can now be expressed conveniently as

$$\mathbf{\Gamma}\mathbf{B}^T = \mathbf{D}, \qquad (9)$$

where $\mathbf{B}^T$ denotes the transposed matrix of $\mathbf{B}$. Equation (9) can be readily solved, for $n = 1$, $p,q = 1,2,3$ and for $n = 2$, $p,q = 1,\dots,5$ and $p < q$, giving 16 analytically solvable regions in the b space, or 16 sets of explicit expressions of b's in terms of the arbitrary amplitudes C_1 and C_2 of the waves, for which Eqs. (4) are analytically solvable. The modulus k of the elliptic functions that express the Lamé functions, which is in the range $0 < k^2 \le 1$ unless otherwise specified, can be regarded as another variable parameter. Treating the amplitudes C_1 and C_2 for x_1 and x_2, the modulus k, the scaling parameter α, and in some cases A_1 and A_2, as variable parameters, the sixteen analytically solvable regions in the b space for Eqs. (4), $N = 2$, are given in (i)–(xvi) in Tables I–III, together with the analytic solutions for x_1 and x_2. Using transformation (6), these are the regions of b's for which Eqs. (1) or (2) have analytic coupled solitary-wave solutions.

These results show surprisingly many analytically solvable regions for the two coupled dynamical equations (4) and for two coupled nonlinear Schrödinger-like equations (1) and (2). The analytically solvable regions given in Tables I and II correspond to analytic solutions given by waves of order $n = 1$, and those given in Table III correspond to waves of order $n = 2$, the order of the Lamé equation. For the analytically solvable regions in Table I, x_1 and x_2 have the same wave form, A_1 must be equal to A_2. The nonlinear coupling parameters b's, on the other hand, as long as they satisfy the equalities and inequalities stated, are quite free to take up rather wide ranges of values. It should be remembered, however, that C_1 and C_2 must be non-negative (for x_1 and

TABLE I. Analytically solvable regions for waves of order 1 and of the same wave form for x_1 and x_2.

	(i)	(ii)	(iii)
x_1	$\sqrt{C_1}\, sn(\alpha t, k)$	$\sqrt{C_1}\, cn(\alpha t, k)$	$\sqrt{C_1}\, dn(\alpha t, k)$
x_2	$\sqrt{C_2}\, sn(\alpha t, k)$	$\sqrt{C_2}\, cn(\alpha t, k)$	$\sqrt{C_2}\, dn(\alpha t, k)$
A_j	$-(1 + k^2)\alpha^2$	$(2k^2 - 1)\alpha^2$	$(2 - k^2)\alpha^2$
	For $b_{11}/b_{21} = b_{12}/b_{22} = 1$,	For $b_{11}/b_{21} = b_{12}/b_{22} = 1$,	For $b_{11}/b_{21} = b_{12}/b_{22} = 1$,
	$b_{11}C_1 + b_{12}C_2 = -2k^2\alpha^2$	$b_{11}C_1 + b_{12}C_2 = 2k^2\alpha^2$	$b_{11}C_1 + b_{12}C_2 = 2\alpha^2$
		For $b_{11} > b_{21}, b_{22} > b_{12}$,	For $b_{11} > b_{21}, b_{22} > b_{12}$,
		$C_1 = 2k^2\alpha^2(b_{22} - b_{12})\Delta^{-1}$,	$C_1 = 2\alpha^2(b_{22} - b_{12})\Delta^{-1}$,
		$C_2 = 2k^2\alpha^2(b_{11} - b_{21})\Delta^{-1}$,	$C_2 = 2\alpha^2(b_{11} - b_{21})\Delta^{-1}$,
		where $\Delta = b_{11}b_{22} - b_{12}b_{21}$	where $\Delta = b_{11}b_{22} - b_{12}b_{21}$

x_2 to be real), and thus at least one of the b's in (i), for example, must be negative. For the analytically solvable regions in Tables II and III, x_1 and x_2 have different (or what we call complementary) wave forms and, for these regions, A_1 need not be equal to A_2. While A_1 and A_2 are free to take up any values for complementary waves of order 1 as shown in (iv)–(vi), they are constrained for complementary waves of order 2 as shown in (vii)–(xvi). For these analytically solvable regions, the nonlinear coupling parameters b's can assume wide ranges of values as indicated. Some have common boundaries at $k^2 = 1$.

It is possible that a similar approach can be used for finding analytically solvable regions for other nonlinearly coupled equations, and that these results can be used as a starting point for discovering a more general way of finding analytically solvable regions and how these regions are linked to the integrable points in nonlinear dynamical problems. As they stand now, the explicit expressions (i)–(xvi) could open up new applications in optical communications. We note that some wave pairs can be in the "mixed" GVD region, i.e., one wave in the normal while the other in the anomalous GVD region, and some wave pairs can be in the normal or anomalous GVD regions for both waves. But the new feature here is that they are not always restricted for use in those regions because, depending on the choice of amplitudes and modulus, the same wave pair can be made to propagate as a solitary wave pair in two optical media of different character. Prospects for experimental applications of these shape-preserving "Jacobian elliptic wave trains" have been greatly enhanced following a recent experimental observation [7] of the evolution of an arbitrarily shaped input optical pulse train to the shape-preserving Jacobian elliptic pulse train for the Maxwell-Bloch equations. If, however, we restrict ourselves to using only aperiodic waves that correspond to $k^2 = 1$, then the analytically solvable regions are reduced in number and size considerably; the aperiodic solitary waves have the forms $\tanh \alpha\xi$ and $\mathrm{sech}\, \alpha\xi$ for waves of order 1 (the well-known dark and bright solitary waves) and have the forms $\mathrm{sech}^2\alpha\xi - \frac{2}{3}$, $\tanh \alpha\xi\, \mathrm{sech}\, \alpha\xi$, and $\mathrm{sech}^2\alpha\xi$ (the red, white, and blue solitary waves [4]) for waves of order two. These aperiodic waves of orders greater than 1 can be multihump solitary waves, and it is interesting to note a recent experimental observation of multihump solitons in a dispersive nonlinear medium [8] and the appearance of two of the three wave forms of order 2 in the theory of incoherent dark solitons [9]. Besides applications in optical communication, another potentially useful application of the results presented in this Letter is in the study of the dynamical stability and creation of solitary waves in multispecies Bose-Einstein condensates [10].

TABLE II. Analytically solvable regions for complementary waves of order 1 for x_1 and x_2.

	(iv)	(v)	(vi)
x_1	$\sqrt{C_1}\, sn(\alpha t, k)$	$\sqrt{C_1}\, sn(\alpha t, k)$	$\sqrt{C_1}\, cn(\alpha t, k)$
x_2	$\sqrt{C_2}\, cn(\alpha t, k)$	$\sqrt{C_2}\, dn(\alpha t, k)$	$\sqrt{C_2}\, dn(\alpha t, k)$
b_{11}	$[A_1 + (1 - k^2)\alpha^2]C_1^{-1}$	$k^2[A_1 - (1 - k^2)\alpha^2]C_1^{-1}$	$-k^2k'^{-2}[A_1 - \alpha^2]C_1^{-1}$
b_{12}	$[A_1 + (1 + k^2)\alpha^2]C_2^{-1}$	$[A_1 + (1 + k^2)\alpha^2]C_2^{-1}$	$k'^{-2}[A_1 + (1 - 2k^2)\alpha^2]C_2^{-1}$
b_{21}	$[A_2 + (1 - 2k^2)\alpha^2]C_1^{-1}$	$k^2[A_2 - (2 - k^2)\alpha^2]C_1^{-1}$	$-k^2k'^{-2}[A_2 + (2 - k^2)\alpha^2]C_1^{-1}$
b_{22}	$[A_2 + \alpha^2]C_2^{-1}$	$[A_2 + k^2\alpha^2]C_2^{-1}$	$k'^{-2}[A_2 - k^2\alpha^2]C_2^{-1}$

TABLE III. Analytically solvable regions for complementary waves of order 2 for x_1 and x_2. $G_\pm$ denotes $1 + k^2 \pm (1 - k^2 + k^4)^{1/2}$.

	(vii)	(viii)
x_1	$\sqrt{C_1}\,[\frac{1}{3}G_- - k^2 sn^2(\alpha t, k)]$	$\sqrt{C_1}\,[\frac{1}{3}G_- - k^2 sn^2(\alpha t, k)]$
x_2	$\sqrt{C_2}\,sn(\alpha t, k)cn(\alpha t, k)$	$\sqrt{C_2}\,sn(\alpha t, k)dn(\alpha t, k)$
b_{11}	$9G_-^{-2}[A_1 + 2G_+\alpha^2]C_1^{-1}$	$9G_-^{-2}[A_1 + 2G_+\alpha^2]C_1^{-1}$
b_{12}	$6k^2G_-^{-1}[A_1 + (2G_+ - G_-)\alpha^2]C_2^{-1}$	$6k^2G_-^{-1}[A_1 + (2G_+ - G_-)\alpha^2]C_2^{-1}$
b_{21}	$9G_-^{-2}[A_2 + (4 + k^2)\alpha^2]C_1^{-1}$	$9G_-^{-2}[A_2 + (1 + 4k^2)\alpha^2]C_1^{-1}$
b_{22}	$6k^2G_-^{-1}[A_2 + (4 + k^2 - G_-)\alpha^2]C_2^{-1}$	$6k^2G_-^{-1}[A_2 + (1 + 4k^2 - G_-)\alpha^2]C_2^{-1}$
A_1	$2\alpha^2(2G_+G_- - 3k^2G_+ - 1)/(3k^2 - 2G_-)$	$2\alpha^2[G_-(2G_+ - G_-) - 3G_+]/(3 - 2G_-)$
A_2	$\alpha^2[2(4 + k^2)G_- - 3k^2(4 + k^2) - 2]/(3k^2 - 2G_-)$	$\alpha^2[2G_-(1 + 4k^2 - G_-) - 3(1 + 4k^2)]/(3 - 2G_-)$
	(ix)	(x)
x_1	$\sqrt{C_1}\,[\frac{1}{3}G_- - k^2 sn^2(\alpha t, k)]$	$\sqrt{C_1}\,[\frac{1}{3}G_- - k^2 sn^2(\alpha t, k)]$
x_2	$\sqrt{C_2}\,cn(\alpha t, k)dn(\alpha t, k)$	$\sqrt{C_2}\,[\frac{1}{3}G_+ - k^2 sn^2(\alpha t, k)]$
b_{11}	$-9G_-^{-1}\Delta^{-1}\{(1 + k^2)A_1 + 2[G_+(1 + k^2) - 3k^2]\alpha^2\}C_1^{-1}$	$-9G_-^{-1}\Delta^{-1}(A_1 + G_+\alpha^2)C_1^{-1}$
b_{12}	$6k^2\Delta^{-1}\{A_1 + (2G_+ - G_-)\alpha^2\}C_2^{-1}$	$9G_+^{-1}\Delta^{-1}\{A_1 + (2G_+ - G_-)\alpha^2\}C_2^{-1}$
b_{21}	$-9G_-^{-1}\Delta^{-1}\{(1 + k^2)A_2 + (1 - 4k^2 + k^4)\alpha^2\}C_1^{-1}$	$-9G_-^{-1}\Delta^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1}$
b_{22}	$6k^2\Delta^{-1}\{A_2 + (1 + k^2 - G_-)\alpha^2\}C_2^{-1}$, where $\Delta = 6k^2 - G_-(1 + k^2)$	$9G_+^{-1}\Delta^{-1}(A_2 + G_-\alpha^2)C_2^{-1}$, where $\Delta = G_+ - G_-$
A_1	$2\alpha^2\{G_-(2G_+ - G_-) - 3[G_+(1 + k^2) - 3k^2]\}/[3(1 + k^2) - 2G_-]$	$\alpha^2(2G_+G_- - G_+^2 - G_-^2)\Delta^{-1}$
A_2	$\alpha^2\{2G_-(1 + k^2 - G_-) - 3(1 - 4k^2 + k^4)\}/[3(1 + k^2) - 2G_-]$	$\alpha^2(-2G_+G_- + G_+^2 + G_-^2)\Delta^{-1}$
	(xi)	(xii)
x_1	$\sqrt{C_1}\,sn(\alpha t, k)cn(\alpha t, k)$	$\sqrt{C_1}\,sn(\alpha t, k)cn(\alpha t, k)$
x_2	$\sqrt{C_2}\,sn(\alpha t, k)dn(\alpha t, k)$	$\sqrt{C_2}\,cn(\alpha t, k)dn(\alpha t, k)$
b_{11}	$6\alpha^2k^4k'^{-2}C_1^{-1}$	$\{(1 + k^2)A_1 + (4 - k^2 + k^4)\alpha^2\}C_1^{-1}$
b_{12}	$-6\alpha^2k^2k'^{-2}C_2^{-1}$	$\{A_1 + (4 + k^2)\alpha^2\}C_2^{-1}$
b_{21}	$6\alpha^2k^4k'^{-2}C_1^{-1}$	$\{(1 + k^2)A_2 + (1 - 4k^2 + k^4)\alpha^2\}C_1^{-1}$
b_{22}	$-6\alpha^2k^2k'^{-2}C_2^{-1}$	$\{A_2 + (1 + k^2)\alpha^2\}C_2^{-1}$
A_1	$-(4 + k^2)\alpha^2$	$(5k^2 - 4)\alpha^2$
A_2	$-(1 + 4k^2)\alpha^2$	$(5k^2 - 1)\alpha^2$
	(xiii)	(xiv)
x_1	$\sqrt{C_1}\,sn(\alpha t, k)cn(\alpha t, k)$	$\sqrt{C_1}\,sn(\alpha t, k)dn(\alpha t, k)$
x_2	$\sqrt{C_2}\,[\frac{1}{3}G_+ - k^2 sn^2(\alpha t, k)]$	$\sqrt{C_2}\,cn(\alpha t, k)dn(\alpha t, k)$
b_{11}	$6k^2G_+^{-1}\{A_1 + (4 + k^2 - G_+)\alpha^2\}C_1^{-1}$	$\{(1 + k^2)A_1 + (1 - k^2 + 4k^4)\alpha^2\}C_1^{-1}$
b_{12}	$9G_+^{-2}\{A_1 + (4 + k^2)\alpha^2\}C_2^{-1}$	$\{A_1 + (1 + 4k^2)\alpha^2\}C_2^{-1}$
b_{21}	$6k^2G_+^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1}$	$\{(1 + k^2)A_2 + (1 - 4k^2 + k^4)\alpha^2\}C_1^{-1}$
b_{22}	$9G_+^{-2}\{A_2 + 2G_-\alpha^2\}C_2^{-1}$	$\{A_2 + (1 + k^2)\alpha^2\}C_2^{-1}$
A_1	$\alpha^2\{3k^2(4 + k^2) - 2G_+(4 + k^2 - G_+)\}/(2G_+ - 3k^2)$	$(5 - 4k^2)\alpha^2$
A_2	$\alpha^2\{6k^2G_- - 2G_+(2G_- - G_+)\}/(2G_+ - 3k^2)$	$(5 - k^2)\alpha^2$
	(xv)	(xvi)
x_1	$\sqrt{C_1}\,sn(\alpha t, k)dn(\alpha t, k)$	$\sqrt{C_1}\,cn(\alpha t, k)dn(\alpha t, k)$
x_2	$\sqrt{C_2}\,[\frac{1}{3}G_+ - k^2 sn^2(\alpha t, k)]$	$\sqrt{C_2}\,[\frac{1}{3}G_+ - k^2 sn^2(\alpha t, k)]$
b_{11}	$6k^2G_+^{-1}\{A_1 + (1 + 4k^2 - G_+)\alpha^2\}C_1^{-1}$	$-6G_+k^2\Delta^{-1}\{A_1 + (1 + k^2 - G_+)\alpha^2\}C_1^{-1}$
b_{12}	$9G_+^{-2}\{A_1 + (1 + 4k^2)\alpha^2\}C_2^{-1}$	$9\Delta^{-1}\{(1 + k^2)A_1 + (1 - 4k^2 + k^4)\alpha^2\}C_2^{-1}$
b_{21}	$6k^2G_+^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1}$	$-6G_+k^2\Delta^{-1}\{A_2 + (2G_- - G_+)\alpha^2\}C_1^{-1}$
b_{22}	$9G_+^{-2}\{A_2 + 2G_-\alpha^2\}C_2^{-1}$	$9\Delta^{-1}\{(1 + k^2)A_2 + [2G_-(1 + k^2) - 6k^2]\alpha^2\}C_2^{-1}$, where $\Delta = (1 + k^2)G_+^2 - 6G_+k^2$
A_1	$\alpha^2\{3(1 + 4k^2) - 2G_+(1 + 4k^2 - G_+)\}/(2G_+ - 3)$	$\alpha^2\{2G_+(1 + k^2 - G_+) - 3(1 - 4k^2 + k^4)\}/[3(1 + k^2) - 2G_+]$
A_2	$\alpha^2\{6G_- - 2G_+(2G_- - G_+)\}/(2G_+ - 3)$	$\alpha^2\{2G_+(2G_- - G_+) - 6[G_-(1 + k^2) - 3k^2]\}/[3(1 + k^2) - 2G_+]$

[1] See, e.g., J. Hietarinta, Phys. Rep. **147**, 87 (1987); A.V. Mikhailov *et al.*, Russ. Math. Surveys **42**, 1 (1987), and references therein.

[2] See, e.g., G.P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, New York, 1995), and references therein.

[3] S.V. Manakov, Sov. Phys. JETP **38**, 248 (1974).

[4] See, e.g., F.T. Hioe, Phys. Rev. E **58**, 6700 (1998), and references therein.

[5] Y. Kajinaga *et al.*, J. Phys. Soc. Jpn. **67**, 1565 (1998).

[6] See, e.g., *Higher Transcendental Functions*, edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1955), Vol. 3, Chap. XV.

[7] J.L. Shultz and G.J. Salamo, Phys. Rev. Lett. **78**, 855 (1997).

[8] M. Mitchell *et al.*, Phys. Rev. Lett. **80**, 4657 (1998).

[9] D.N. Christodoulides and T.H. Coskun, Phys. Rev. Lett. **80**, 5113 (1998).

[10] C.K. Law *et al.*, Phys. Rev. Lett. **79**, 3105 (1997); R. Dum *et al.*, *ibid.* **80**, 2972 (1998).