Quantum Lévy Processes and Fractional Kinetics

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Exotic stochastic processes are shown to emerge in the quantum evolution of complex systems. Using influence function techniques, we consider the dynamics of a system coupled to a chaotic subsystem described through random matrix theory. We find that the reduced density matrix can display dynamics given by Lévy stable laws. The classical limit of these dynamics can be related to fractional kinetic equations. In particular, we derive a fractional extension of Kramers equation. [S0031-9007(99)08391-X]

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Whether one studies deterministic Hamiltonian or dissipative systems, one finds that transport in chaotic systems often resembles some type of stochastic process. The dynamics of such systems leads to a rich spectrum of behaviors, from enhanced diffusion such as tracer diffusion in flows and turbulent diffusion in the atmosphere, to dispersive diffusion [1]. Much effort has been spent in recent years to understand such classical stochastic processes in chaotic systems, leading to the development of approaches ranging from fractional kinetic equations [2–4] and Lévy flights [5], to random walks in random environments [5,6] and stochastic webs [7]. One of the common features to all of these is the use of Lévy stable laws [8]. It was shown by Lévy [9], in studies of extensions of the central limit theorem, that a continuous class of non-Gaussian processes satisfy the same fundamental equation that gives rise to the theory of Gaussian processes, namely, the Chapman-Kolmogorov equation for the conditional probability $P(q,q',t)$:

$$P(q,q';t) = \int dq''\, P(q,q'',t-t'')P(q'',q',t'') . \quad (1)$$

The standard solution, $P(q,q',t) = \exp[-(q-q')^2/4Dt]/(4\pi Dt)^{3/2}$, gives rise to the Gaussian processes and the usual form of the Fokker-Planck equation. The general solutions of Lévy provide a way to generalize Brownian motion.

The non-Gaussian processes which satisfy (1) are known as Lévy stable laws, and have the form

$$P(q,t) = \mathcal{L}_\alpha^A(q) = \frac{1}{2\pi}\int \exp\{ikq - A|k|^\alpha\}\,dk , \quad (2)$$

where $0 < \alpha \le 2$ and $A \propto t$. The case $\alpha = 2$ corresponds to Gaussian processes. The Lévy distributions $\mathcal{L}_\alpha^A(q)$ satisfy the scaling relation

$$\mathcal{L}_\alpha^A(q) = A^{-1/\alpha}\mathcal{L}_\alpha^1(qA^{-1/\alpha}) , \quad (3)$$

where for $A = 1$ we drop the superscript: $\mathcal{L}_\alpha^1(x) = \mathcal{L}_\alpha(x)$. For $\alpha < 2$, these distributions are characterized by infinite second moments, as one can easily infer from the asymptotic behavior for $q \to \pm\infty$ [5],

$$\mathcal{L}_\alpha^A(q) \approx \Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,\frac{\alpha A}{|q|^{\alpha+1}} . \quad (4)$$

These non-Gaussian processes can be related to anomalous transport in a variety of (classical) physical systems [6], as well as to classically chaotic systems. We have recently shown that turbulent diffusion can also arise in the time evolution of complex quantum systems [10]. Here we find that a general form of quantum chaotic backgrounds can give rise to quantum evolution characterized by Lévy distributions. Further, we can connect, in the semiclassical limit, such processes to fractional kinetic theory, which was initially introduced as a phenomenological approach to classical anomalous diffusion.

We study the problem of a particle coupled to a chaotic environment, quantum mechanically. It has been realized in recent years that the quantum counterpart of chaos is random matrix theory. For systems with time-reversal symmetry, the random matrices are real symmetric. In this Letter we will examine the class of quantum dynamic processes, which can be realized through the interaction of a particle with a random matrix background. In contrast to the Caldeira-Leggett approach [11], we assume from the outset that the background is chaotic, and not necessarily thermal. We denote the coordinates of the background by (x,p) and that of the test particle by (X,P). The Hamiltonian for the background plus interaction is taken to have the following form:

$$H_b = h_0(x,p) + h_1(X,x,p) . \quad (5)$$

In the basis of (many-body) eigenstates of h_0, $h_0|n\rangle = \varepsilon_n|n\rangle$ $(n = 1,\dots,N)$, we define the matrix of H_b as

$$[H_b]_{ij} = \varepsilon_i\delta_{ij} + [h_1(X)]_{ij} . \quad (6)$$

It is convenient for calculations to choose an average level density as $\rho(\varepsilon) = \rho_0\exp(\beta\varepsilon)$. For a background with constant average level density, $\beta = 0$, while for a general many-body system, $\beta > 0$. The chaotic properties of the

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interaction of the background with the particle are incorporated into the correlation function (second cumulant):

$$\langle\langle [h_1(X)]_{ij}[h_1(Y)]_{kl}\rangle\rangle = \mathcal{G}_{ij}(X - Y)\Delta_{ijkl}\,. \qquad (7)$$

Here $\Delta_{ijkl} = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$, and all other cumulants vanish. In our analysis, the integration over the chaotic part, given by $h_1(X)$, is defined through a Gaussian measure for parametric random matrices [12],

$$P[h_1(X)] \propto \exp\left\{-\frac{1}{2}\int dX\,dY \times \mathrm{Tr}[h_1(X)\mathcal{G}^{-1}(X - Y)h_1(Y)]\right\}. \qquad (8)$$

The character of the interaction of the background with the test particle is incorporated into the correlation function $\mathcal{G}(X - Y)$, for which we use the form [13–15]

$$\mathcal{G}_{ij}(X) = \frac{\Gamma^{\downarrow}}{2\pi\sqrt{\rho(\varepsilon_i)\rho(\varepsilon_j)}}\exp\left[-\frac{(\varepsilon_i - \varepsilon_j)^2}{2\kappa_0^2}\right]G\left(\frac{X}{X_0}\right). \qquad (9)$$

This describes a parametric, banded, random matrix where the strength of matrix elements decreases with increasing level density. Here $G(x) = G(-x) = G^*(x) \le 1$, $G(0) = 1$, and the spreading width $\Gamma^{\downarrow}$, κ_0 [linked with the effective band width $N_0 \approx \kappa_0\rho(\varepsilon)$], and the correlation length X_0 are characteristics of the background.

In order for the measure (8) to be positive definite G must not decorrelate faster than a Gaussian [12]:

$$G(x) \simeq 1 - |x|^{\alpha} + \ldots, \qquad \alpha \in (0,2]. \qquad (10)$$

As the position X of the slow particle changes, the instantaneous energy levels $E_n(X)$ of $[H_b(X)]_{ij}$ change. Using the above measure, the average fluctuations are

$$\langle [E_n(X) - E_n(Y)]^2\rangle = D_{\alpha}|X - Y|^{\alpha}. \qquad (11)$$

The energy-spacing fluctuations have a behavior that is similar to a Lévy process characterized by the diffusion constant D_{α}. The character of these fluctuations in the eigenvalues $E_n(X)$, indicated by α, will be seen to be related to Lévy distributions, which describe the time evolution of the density matrix for a particle evolving in this chaotic bath.

To develop the dynamical evolution of a free particle evolving in the presence of a chaotic background, we take the Hamiltonian of the form

$$H_{ij}(X,P) = \delta_{ij}\left[\frac{P^2}{2M} + U(X)\right] + H_{b,ij}(X)\,. \qquad (12)$$

The correlated, random-matrix bath can be integrated out in an influence functional formalism [14]. For our purposes, the $o(\beta)$ action is sufficient, as well as a weak coupling of the particle to the bath. In this case the effective equation for the density matrix of the test particle has the form

$$i\hbar\partial_t\rho(X,Y,t) = \left\{\frac{P_X^2}{2M} - \frac{P_Y^2}{2M} + U(X) - U(Y) - \frac{\beta\Gamma^{\downarrow}\hbar}{4X_0M}G'\left(\frac{X - Y}{X_0}\right)(P_X - P_Y) + i\Gamma^{\downarrow}\left[G\left(\frac{X - Y}{X_0}\right) - 1\right]\right\}\rho(X,Y,t), \qquad (13)$$

where in weak coupling $G(x) = 1 - |x|^{\alpha}$ and $G'(x)$ above represents $-\alpha\ \mathrm{sgn}(x)\,|x|^{\alpha-1}$.

Consider first a test particle in the absence of an external field and interacting with a background with constant average level density [$U(X) = 0$ and $\beta = 0$]. This evolution equation can be solved by passing to the coordinates $r = (X + X')/2$, $s = X - X'$. In these variables, the density matrix has the form

$$\rho(r,s,t) = \int dr' \int \frac{dk}{2\pi\hbar}\rho_0\left(r', s - \frac{kt}{m}\right) \times \exp\left\{\frac{ik(r - r')}{\hbar} + \frac{\Gamma^{\downarrow}M}{\hbar k}\int_{s-kt/M}^{s} ds' \times\left[G\left(\frac{s'}{X_0}\right) - 1\right]\right\}. \qquad (14)$$

An initial wave packet, $\psi_0(X) = \exp(-X^2/4\sigma^2)/(2\pi\sigma^2)^{1/4}$, provides an initial density matrix $\rho_0(r,s') = (1/\sqrt{2\pi\sigma^2})\exp[-(4r^2 + s^2)/8\sigma^2]$.

For the diffusive dynamics of the test particle, we are interested in the diagonal component of the density matrix $\rho(X,X,t) = \rho(r,s = 0,t)$,

$$\rho(r,0,t) = \int\int \frac{dr'dk}{2\pi\hbar}\rho_0(r', -kT/M) \times \exp\left[\frac{ik(r - r')}{\hbar} - \int_{-kt/M}^{0} ds' \times \frac{M\Gamma^{\downarrow}}{k\hbar}\left|\frac{s'}{X_0}\right|^{\alpha}\right] \qquad (15)$$

$$= \int \frac{dk}{2\pi\hbar}\exp\left\{-k^2\left[\frac{\sigma^2}{2\hbar^2} + \frac{t^2}{8M\sigma^2}\right] - \frac{\Gamma^{\downarrow}t^{\alpha+1}}{(\alpha + 1)\hbar(MX_0)^{\alpha}}|k|^{\alpha} + ik\frac{r}{\hbar}\right\}. \qquad (16)$$

$\rho(X,X,t)$ is nothing more than the spatial probability distribution $P(X,t)$ for the process. We can now express it in terms of a convolution of Lévy distributions,

$$\rho(X,X,t) = \int dX'\,\mathcal{L}_{\alpha}^{a(t)}(X')\mathcal{L}_2^{b(t)}(X - X'), \qquad (17)$$

where

$$a(t) = \frac{\Gamma^{\downarrow}}{(\alpha + 1)\hbar} \left(\frac{\hbar}{M X_0} \right)^{\alpha} t^{\alpha+1}, \tag{18}$$

$$b(t) = \frac{\sigma^2}{2} + \frac{\hbar^2}{8M^2\sigma^2}\, t^2. \tag{19}$$

As both functions in the integrand of Eq. (17) are positive definite, the spatial probability $P(X,t)$ is also positive definite. Notice that the restriction of $0 < \alpha \leq 2$, which came from the short-distance statistical correlations in (11) and the requirement of a positive definite statistical measure, is also the necessary requirement on the Lévy distribution to keep the resulting time evolution positive definite. Hence the character of the *short-distance* fluctuations is directly responsible for the *long-time* behavior of the quantum system.

Consider now the short-time and long-time behavior of the dynamics. For $1 < \alpha < 2$, in the limit of long times, we expect the $t^{\alpha+1}$ term to dominate over t^2 in (16), so that the density asymptotically approaches a Lévy distribution,

$$\rho(X,X,t) \to a(t)^{-1/\alpha} \mathcal{L}_\alpha[a(t)^{-1/\alpha} X], \tag{20}$$

while for very short times, the Gaussian process is the dominant behavior,

$$\rho(X,X,t) \to \frac{\sqrt{2}}{\sigma} \mathcal{L}_2\left(\frac{\sqrt{2}}{\sigma} X \right). \tag{21}$$

Specifically, in Eq. (16) the $|k|^\alpha$ term in the exponent dominates in the long-time limit only for momenta $k < k_c$, where

$$k_c = \left[\frac{a(t)}{b(t)} \right]^{1/(2-\alpha)} \propto t^{(\alpha-1)/(2-\alpha)}. \tag{22}$$

For the special case of $\alpha = 2$, the result is Gaussian, but the dynamics can be anomalous, as one can have turbulentlike diffusion of the type $\langle X^2 \rangle \sim t^3$ [10]. When the level density of the background is not constant, $\beta > 0$ and $\alpha = 2$, one can recover Brownian diffusion [15]. For general α and $\beta > 0$, however, the results are not yet known. For the range $0 < \alpha < 1$, the long-time behavior approaches a Gaussian process. At short times, the dynamics is influenced by $\mathcal{L}_\alpha$, and there is a crossover from short time Lévy dynamics to normal Gaussian expansion of the wave packet. In both cases, however, $1 < \alpha < 2$ and $0 < \alpha < 1$, the second spatial moments are strictly speaking divergent. One should also note that neither $a(t)$ nor $b(t)$ are linear in time; even though the dynamics has the character of a Lévy process, it is not a Lévy stable law.

Efforts to understand unusual stochastic behaviors of dynamical systems has led to the development of extensions of the Fokker-Planck (FP) equation [2–4]. These are phenomenological fractional kinetic equations (restricted to one dimension) in which certain derivatives are replaced by derivatives of "fractional" order [16]. Such approaches have also found applications in a wide range of problems from turbulence to diffusion in porous or viscoelastic media [17]. We can now explore the type of stochastic process, which emerges in the classical limit of our quantum Lévy processes, and the connection to multidimensional fractional kinetic theory.

Typically, the phenomenological fractional FP equation has the form

$$\frac{\partial^\delta P(Q,t)}{\partial t^\beta} = \frac{\partial^\mu}{\partial(-Q)^\mu} [A(Q)P(Q,t)] + \frac{1}{2} \frac{\partial^{2\nu}}{\partial(-Q)^{2\nu}} [B(Q)P(Q,t)], \tag{23}$$

where $\mu = \nu = 1$ in Ref. [4], $\mu = \nu$ in Ref. [3], and $\delta = \nu = 1$ in Ref. [2]. Here the symbol $\partial^\mu/\partial x^\mu$ represents the Riemann-Liouville fractional derivative [16], except for Ref. [2], where it represents the Fourier transform of $-k^\mu$. This equation, while formally constructed, is phenomenological. It is defined to reproduce anomalous diffusion through scaling formulas such as $Q^2 \sim t^\gamma$, where γ is a function of δ, μ, and ν. A few points should be made here. Generally, the coefficients A and B are defined as limits whose existence is postulated but not known. Further, either the form of the fractional derivatives is taken to provide this scaling law, or power law noise is chosen to obtain them [2,6]. Such dynamics can then be related to Lévy processes [1]. Finally, the extension of these equations to phase space become tenuous, since it is not clear how to include momentum. Note only is it unclear if one should take fractional derivatives with respect to coordinates, momenta, or both, but the existence of the corresponding coefficients $A, B, \ldots$ is unknown. Through our transport equation, we can provide a microscopic interpretation of these coefficients as well as a systematic manner to construct a fractional kinetic equation is phase space, whose quantum limit results in Lévy processes.

To obtain a classical transport equation, we construct the Wigner transform $f(Q,P,t)$ of the density matrix $\rho(X,Y,t)$ as

$$f = \int \frac{dR}{2\pi\hbar} \exp\left(-\frac{iPR}{\hbar} \right) \rho\left(Q + \frac{R}{2}, Q - \frac{R}{2}, t \right). \tag{24}$$

Applying this to our evolution equation, taking the leading order terms in $\hbar$, we find

$$\frac{\partial f}{\partial t} = \int \frac{dR}{2i\pi\hbar^2} \exp\left(-\frac{iPR}{\hbar} \right) \left\{ -\frac{\hbar^2}{2M} \partial_Q \partial_R + U\left(Q + \frac{R}{2} \right) - U\left(Q - \frac{R}{2} \right) - i\Gamma^{\downarrow} \left| \frac{R}{X_0} \right|^\alpha + i\gamma\hbar X_0 \alpha \,\mathrm{sgn}(R) \left| \frac{R}{X_0} \right|^{\alpha-1} \partial_R \right\} \rho\left(Q + \frac{R}{2}, Q - \frac{R}{2}, t \right). \tag{25}$$

This leads naturally to the Reisz fractional integro-differential operator. This operator, applied to a function $f(P)$, is defined as [16]

$$(-\Delta_P)^{\alpha/2} f = \mathcal{F}^{-1}|X|^\alpha \mathcal{F} f\,, \tag{26}$$

where Δ_P is the Laplacian (in our case with respect to the momentum P), and $\mathcal{F}$ represents a Fourier transform. (This operator is distinct from that proposed in [2] which did not have the absolute value, and from [3], which uses the Riemann-Liouville form of this operator. The Reisz operator is defined as a fractional integral for Re $\alpha < 0$ and as a fractional derivative for Re $\alpha > 0$ through analytic continuation.) It is convenient to define the operator $D_P^\alpha = (-i/\hbar)^\alpha(-\Delta_P)^{\alpha/2}$, since $D_P^2[f] = \partial^2 f/\partial P^2$. Then the classical limit of our quantum Lévy process gives rise to a fractional extension of Kramers equation:

$$\frac{\partial f(Q,P,t)}{\partial t} + \frac{P}{M}\frac{\partial f(Q,P,t)}{\partial Q} - \frac{\partial U(Q)}{\partial Q}\frac{\partial f(Q,P,t)}{\partial P} = \gamma_\alpha\left\{\overline{D}_P^{\alpha-1}[Pf(Q,P,t)] - \frac{2TM}{\alpha\hbar^2}(i\hbar)^\alpha D_P^\alpha[f(Q,P,t)]\right\}, \tag{27}$$

where $T = 1/\beta$ is the temperature and the operator $\overline{D}_P^\alpha = (-i/\hbar)^\alpha \mathcal{F}^{-1}\,\mathrm{sgn}(X)\,|X|^\alpha\mathcal{F}$, with the property $\overline{D}_P^1[Pf] = \partial(Pf)/\partial P$. The generalized friction coefficient is given by

$$\gamma_\alpha = \frac{\beta\Gamma^\downarrow\hbar\alpha}{2MX_0^\alpha}\,. \tag{28}$$

For $\alpha = 2$ we recover Kramers equation [18]. What we see is that it is not the coordinates which acquire the fractional character, as usually assumed, but the momenta. Because the coupling to the background is not momentum dependent, the correlation function $G(X)$ results only in fractional derivatives with respect to momenta. This can be traced back to the nature of the chaotic correlations in Eq. (13). Further, these processes, related to Lévy processes, do not require the introduction of fractional time derivatives. We note here that our transport theory has a consistent classical limit for all of these transport coefficients only when they remain finite as $\hbar \to 0$. This requires in turn that the parameters of our quantum theory cannot remain constant as $\hbar \to 0$, if we are to recover a well defined classical transport. Finally, we observe that this approach provides finite coefficients D_{QQ}, D_{PP}, D_{QP}, and so forth (e.g., $A, B, \ldots$) for a fractional kinetic equation in phase space.

We have shown that the quantum evolution of a wave packet in a chaotic environment can lead to reduced density matrices which behave as Lévy processes. The short-distance energy fluctuations of the background, which are characterized by a parameter $\alpha \in (0,2]$, are found to be precisely related to the quantum time evolution with a Lévy process of the same character α. For $\alpha = 2$ one has Gaussian processes, which can display normal to turbulentlike diffusion or Brownian diffusion ($\beta > 0$), while for $\alpha = 1$ one has the dynamics of the Dyson process. The general quantum evolution of a wave packet displays a crossover between Gaussian and Lévy dynamics. In passing to the classical limit of this behavior, we find that the dynamical evolution results in a fractional kinetic equation, which is a generalization of Kramers equation. For $\alpha = 2$ Kramers theory is recovered. This approach provides a means to develop fractional kinetic theory in more than one dimension, since the expansion coefficients are determined from the microscopic theory. It also provides the possibility to explore the connections between quantum and classical transport in chaotic systems, as well as the links between chaos, quantum statistical fluctuations, Lévy processes, and classical fractional dynamics.

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