A Solution of the Odderon Problem

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The intercept of the odderon trajectory is derived by finding the spectrum of the second integral of motion of the three Reggeon system in high energy QCD. When combined with an earlier solution of the appropriate Baxter equation, this leads to the determination of the low lying states of that system. In particular, the energy of the lowest state gives the intercept of the odderon $\alpha_O(0) = 1 - 0.2472\alpha_s N_c/\pi$. [S0031-9007(98)08354-9]

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One of the still unsolved problems of perturbative QCD is the behavior of the theory in the Regge limit. The intercept of the Pomeron trajectory, the BFKL Pomeron, has been derived in the classic works of Balitskii, Fadin, Kuraev, and Lipatov [1]. The next natural step was to find the intercept of the odderon trajectory, which, however, turned out to be very difficult [2–4].

Important progress was made by Lipatov, Faddeev, and Korchemsky [5,6], who reduced the problem to the solution of a functional equation—the Baxter equation—for physical values of the two relevant constants of motion (q_2 and q_3) of the system of three Reggeized gluons. Various approximation techniques [7–9] for solving the Baxter equation have been used, and in our previous work [10,11] an exact method of constructing a solution for general q_2 and q_3 was developed. However, while the eigenvalues of $\hat{q}_2$ are known, the spectrum of $\hat{q}_3$ remained unavailable, apart from asymptotic results of [7,9]. In this Letter we report on the solution of the eigenproblem of $\hat{q}_3$ which removes the last obstacle in deriving the numerical value of α_O in the leading logarithmic approximation.

The intercept of the odderon trajectory is given by

$$\alpha_O(0) = 1 + \frac{\alpha_s N_c}{4\pi}\left[\epsilon_3(h, q_3) + \overline{\epsilon}_3(\overline{h}, \overline{q}_3)\right], \quad (1)$$

where ϵ_3 and $\overline{\epsilon}_3$ are, respectively, the largest eigenvalues of the three Reggeon Hamiltonian and its antiholomorphic counterpart [6]. The conformal weight h parametrizes the eigenvalues of the Casimir operator $\hat{q}_2$,

$$q_2 = h(1-h), \qquad h = \tfrac{1}{2}(1+m) - i\nu, \qquad m \in Z,\ \nu \in R. \quad (2)$$

Analogous formulas hold for the antiholomorphic sector with $\overline{h} = (1-m)/2 - i\nu$ [6]. After an explicit expression for the energies $\epsilon_e(h, q_3)$ $[\overline{\epsilon}_3(\overline{h}, \overline{q}_3)]$ was derived [10], the only unknown ingredient was the quantization of q_3.

The eigenproblem of the $\hat{q}_3$ operator was formulated in general terms by Lipatov [3]. However, the quantitative solution was lacking due to the complicated and indirect way in which the boundary conditions enter and fix the spectrum. To begin, we quote a general form of the wave function of the compound state of three Reggeized gluons,

$$f(\rho, \overline{\rho}) = \sum_{r,s} c_{rs} f_r(\rho) \overline{f}_s(\overline{\rho}), \quad (3)$$

where $\rho = \{\rho_1, \rho_2, \rho_3\}$ denotes three transverse coordinates in the complex number representation $\rho_k = x_k + iy_k$, $k = 1, 2, 3$. All other quantum numbers are collectively denoted by the index r. Similarly in the antiholomorphic sector $\overline{\rho}_k = x_k - iy_k$. Since the integrals of motion $\hat{q}_2, \hat{q}_3$ commute with the Hamiltonian, one chooses $f(\rho)$ as a simultaneous eigenfunction of these operators.

With the conformally covariant ansatz, $z = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}$, $\mu = h/3$, $\rho_{ij} = \rho_i - \rho_j$, $i, j = 0, 1, \ldots, 3$ (ρ_0 being the center of mass of the system),

$$f_{\rho_0, q_2, q_3}(\rho) = \left(\frac{\rho_{12}\rho_{13}\rho_{23}}{\rho_{10}^2\rho_{20}^2\rho_{30}^2}\right)^{\mu} \Phi^{(h,q_3)}(z), \quad (4)$$

the eigenequation, $\hat{q}_3 f = q_3 f$, reads, in terms of Φ, and at fixed q_2 given by (2),

$$a(z)\frac{d^3}{dz^3}\Phi(z) + b(z)\frac{d^2}{dz^2}\Phi(z) + c(z)\frac{d}{dz}\Phi(z) + d(z)\Phi(z) = 0, \quad (5)$$

where

$$\begin{aligned} a(z) &= z^3(1-z)^3 = \sum_{i=0}^{3} a_i z^{i+3}, \\ b(z) &= 2z^2(1-z)^2(1-2z) = \sum_{i=0}^{3} b_i z^{i+2}, \\ c(z) &= z(z-1)\left[z(z-1)(3\mu+2)(\mu-1) + 3\mu^2 - \mu\right] = \sum_{i=0}^{3} c_i z^{i+1}, \\ d(z) &= \mu^2(1-\mu)(z+1)(z-2)(2z-1) - iq_3 z(1-z) = \sum_{i=0}^{3} d_i z^i. \end{aligned}$$

This is a third order linear differential equation with the three regular singular points at $z = 0$, 1, and ∞, introduced by Lipatov in [3] and investigated in [12] in a slightly

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different form. We will solve this equation by standard methods and identify proper boundary conditions which lead to the quantization of q_3. To this end we first construct a fundamental set of three linearly independent solutions $\vec{u}^{(0)}(z) = (u_1^{(0)}(z), u_2^{(0)}(z), u_3^{(0)}(z))$, around $z = 0$.

$$u_1^{(0)}(z) = z^{s_1} \sum_{n=0}^{\infty} f_n^{(1)} z^n, \qquad u_2^{(0)}(z) = z^{s_2} \sum_{n=0}^{\infty} f_n^{(2)} z^n, \tag{6}$$

$$u_3^{(0)}(z) = \ln(z) u_2^{(0)}(z) + z^{s_3} \sum_{n=0}^{\infty} r_n z^n,$$

where $s_1 = 2h/3$, $s_2 = -h/3 + 1$, $s_3 = -h/3$, and the coefficients of the expansions are determined by the following recursion relations:

$$\begin{aligned} f_n^{(i)} &= -\sum_{m=1}^{3} f_{n-m}^{(i)} \lambda_m(s_i + n)/\lambda_0(s_i + n), \\ f_0^{(i)} &= 1, \qquad f_{-j}^{(i)} = 0, \qquad i = 1,2,\ j > 0, \\ \lambda_m(x) &= a_m(x - m)(x - m - 1)(x - m - 2) \\ &\quad + b_m(x - m)(x - m - 1) \\ &\quad + c_m(x - m) + d_m, \end{aligned} \tag{7}$$

and for the logarithmic solution

$$r_n = -\left(p_{n-1} + \sum_{m=1}^{3} r_{n-m} \lambda_m(s_3 + n) \right) \Big/ \lambda_0(s_3 + n), \qquad r_1 = 1, \qquad r_0 = -p_0/\lambda_1(s_3 + 1), \qquad r_{-1} = 0, \tag{8}$$

$$p_n = \sum_{m=0}^{3} f_{n-m}^{(2)} \gamma_m(s_2 + n), \qquad \gamma_m(x) = a_m[3(x - m)(x - m - 2) + 2] + b_m[2(x - m) - 1] + c_m.$$

There are three physical conditions which our solution should satisfy. The complete *wave function* of the compound system (3) must be (i) single valued in the whole transverse plane of Reggeon coordinates, (ii) invariant under any permutation of Reggeon coordinates (Bose symmetry), and (iii) normalizable with respect to the scalar product $\langle f \,|\, g \rangle = \int d\rho\, d\overline{\rho}\, f^*(\rho, \overline{\rho})\, g(\rho, \overline{\rho}) / \prod_{k=1}^{k=3} |\rho_{kk+1}|^2$, $\rho_4 \equiv \rho_1$, $d\rho = d\rho_1 d\rho_2 d\rho_3$. Together with the analyticity of the solution $\Phi(z)$ these conditions unambiguously determine the spectrum of $\hat{q}_3$.

The series in Eq. (6) is convergent in the unit circle R_0 around $z = 0$, and therefore it determines uniquely the analytic continuation $\vec{u}^{(0)}(z)$ to the cut complex plane. To achieve this continuation in practice we construct two other fundamental sets of solutions $\vec{u}^{(1)}(z)$ and $\vec{u}^{(\infty)}(z)$ around $z = 1$ and $z = \infty$. This could be done analogously to Eqs. (6)–(8), for the transformed equation, however, because of the symmetry under permutations of Reggeon coordinates, one can partly satisfy condition (ii) by a proper choice of these bases. Since under even permutations $(1,2,3) \to (3,1,2)$, $z \to 1 - 1/z$, and $(1,2,3) \to (2,3,1)$, $z \to 1/(1 - z)$, we define

$$\vec{u}^{(1)}(z) = \vec{u}^{(0)}\left(1 - \frac{1}{z}\right), \qquad \vec{u}^{(\infty)}(z) = \vec{u}^{(0)}\left(\frac{1}{1 - z}\right), \tag{9}$$

These series are convergent in the regions R_1: $\mathrm{Re}(z) > 1/2$ and R_∞: $|1 - z| > 1$, respectively. Analytic continuation is realized by the transition matrices,

$$u_i^{(0)}(z) = \Gamma_{ij} u_j^{(1)}(z), \qquad u_i^{(1)}(z) = \Omega_{ij} u_j^{(\infty)}(z), \tag{10}$$

which depend on h and q_3 only. They contain the full information about the system, in particular, about its spectrum. In practice we calculate transition matrices as the solutions of the systems of 3×3 algebraic equations (10) written at some judiciously chosen point $z = \zeta$. For example,

$$\Gamma_{ik} = \frac{W_{ik}}{W(\vec{u}^{(1)}(\zeta))}, \qquad W_{ik} = W(u_k^{(1)} \to u_i^{(0)}), \tag{11}$$

where $W(\vec{u}^{(1)}(\zeta))$ is the Wroński determinant of the fundamental solutions $\vec{u}^{(1)}(\zeta)$. As long as ζ is in the intersection of the convergence regions R_0 and R_1, the matrix elements obtained from Eq. (11) are independent of ζ provided enough terms in the series (6) are included.

Finally we implement the uniqueness constraints (i). It is crucial to observe that requiring single valuedness in the holomorphic and antiholomorphic sectors separately gives a too strong condition and is, in fact, not necessary. Even though the Hamiltonians in both sectors commute, proper boundary conditions should be formulated only for the wave function of the whole system (3). We therefore define a general bilinear form [the power prefactors in (4) are irrelevant for this discussion]

$$\Psi_{h,\overline{h},q_3,\overline{q}_3}(z, \overline{z}) = \vec{\overline{u}}^{(1)}(\overline{z})^T A^{(1)} \vec{u}^{(1)}(z), \tag{12}$$

and demand its uniqueness in the whole transverse plane. The compound function (12) has nine free parameters. (We thank Gregory Korchemsky for the discussion on that point.) By inspecting Eqs. (6) we see that the most general choice of coefficients, consistent with the uniqueness of the wave function in the neighborhood of $z = 1$, is

$$A^{(1)} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & 0 \end{pmatrix}, \tag{13}$$

which has the freedom of three parameters. Incidentally this form also guarantees the normalizability of the wave function. Rewriting the wave function in terms of other bases (9) around $z = 0$ and $z = \infty$ gives other coefficient matrices,

$$A^{(0)} = (\overline{\Gamma}^{-1})^T A^{(1)} \Gamma^{-1}, \qquad A^{(\infty)} = \overline{\Omega}^T A^{(1)} \Omega. \tag{14}$$

Now, uniqueness in the whole transverse plane requires that the transformed matrices [Eq. (14)] have the same

form as (13), with possibly different coefficients. Therefore we require

$$A_{12}^{(0)} = A_{13}^{(0)} = A_{33}^{(0)} = A_{21}^{(0)} = A_{31}^{(0)} = 0\,, \qquad (15)$$

$$A_{12}^{(\infty)} = A_{13}^{(\infty)} = A_{33}^{(\infty)} = A_{21}^{(\infty)} = A_{31}^{(\infty)} = 0\,. \qquad (16)$$

In fact, only one of these sets is sufficient, as can be seen from the following topological argument. Any possible cut in the domain of the full wave function has to begin and end at the singular points of the equation, i.e., at 0, 1, or at ∞. Therefore eliminating two of these points guarantees that there is no cut beginning at the third one. [We have checked for completeness that indeed our numerical solutions of Eqs. (15) and (16) coincide.] Equations (15) and (16) are linear homogeneous equations for the coefficients α, β, and γ. The condition of the existence of a nonzero solution of Eqs. (15) or (16) provides the quantization of q_3 and $\overline{q}_3$ that we looked for. It can conveniently be written as

$$\mathcal{B}_U \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0 \quad \text{and} \quad \mathcal{B}_L \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0\,, \qquad (17)$$

where the rows of the matrix $\mathcal{B}_U$ ($\mathcal{B}_L$) are the coefficients of α, β, and γ in, e.g., $A_{12}^{(\infty)}$, $A_{13}^{(\infty)}$, and $A_{33}^{(\infty)}$ ($A_{21}^{(\infty)}$, $A_{31}^{(\infty)}$, and $A_{33}^{(\infty)}$). Explicitly

$$\mathcal{B}_U = \begin{pmatrix} \overline{\Omega}_{11}\Omega_{12} & \overline{\Omega}_{21}\Omega_{22} & \overline{\Omega}_{21}\Omega_{32} + \overline{\Omega}_{31}\Omega_{22} \\ \overline{\Omega}_{11}\Omega_{13} & \overline{\Omega}_{21}\Omega_{23} & \overline{\Omega}_{21}\Omega_{33} + \overline{\Omega}_{31}\Omega_{23} \\ \overline{\Omega}_{13}\Omega_{13} & \overline{\Omega}_{23}\Omega_{23} & \overline{\Omega}_{23}\Omega_{33} + \overline{\Omega}_{33}\Omega_{23} \end{pmatrix}. \qquad (18)$$

Figure 1 shows the, suitably transformed, absolute value of the determinant of $\mathcal{B}_U$ as a function of q_3 along the imaginary axis in the complex q_3 plane. The eigenvalue of the first Casimir operator q_2 is fixed to $q_2 = \overline{q}_2 = 1/4$ which corresponds to the lowest representation of the $SL(2, C)$. All formulas in the antiholomorphic sector are the same with $\overline{q}_3 = q_3^{\star}$, with $\star$ denoting a complex conjugate [6]. Clearly a set of discrete q_3 values exists where the first condition (17) is satisfied. Imposing the second condition eliminates half of the candidates. In addition, a discrete series of real solutions of (17) exists. Both groups lead to single-valued wave functions, however, the condition of Bose symmetry singles out the imaginary q_3 only. Namely, for purely imaginary eigenvalues, the matrices $A^{(0)}$, $A^{(1)}$, and $A^{(\infty)}$ coincide. This, together with the definition of the basis (9), guarantees the invariance of the wave function under even permutations. In order to implement a full Bose symmetry it suffices to take the complete wave function as

$$\Psi(z, \overline{z}) = \Psi_{1/2,1/2,q_3,\overline{q}_3}(z, \overline{z}) + \Psi_{1/2,1/2,q_3,\overline{q}_3}\left(\frac{z}{z-1}, \frac{\overline{z}}{\overline{z}-1}\right), \qquad (19)$$

The second term is just the wave function in the $(-q_3, -\overline{q}_3)$ sector which, due to the degeneracy $q_3 \leftrightarrow -q_3$, obeys the general structure (3). On the other hand, for real eigenvalues q_3, the matrices $A^{(0)}$, $A^{(1)}$, and $A^{(\infty)}$ differ by a phase factor of $e^{2\pi i/3}$, and the only symmetric solution is identical with zero.

For q_3 lying outside of the real and imaginary axes *both* constraints (17) cannot be satisfied simultaneously. There also exists a solution of (17) with $q_3 = 0$. It follows from (8) that the coefficients r_n diverge like $1/q_3$, hence, after rescaling $u_3^{(0)} \to q_3 u_3^{(0)}$, the new solution is free from the logarithmic term at $q_3 = 0$. In fact, in this case the series (7) can be summed and the well-known Pomeron wave functions are recovered. Together with the prefactors as in (4) they depend only on two out of the three coordinates; therefore, they are not normalizable and should be excluded [6]. Moreover, at (and only at) $q_3 = 0$ due to the vanishing of the logarithmic term, other than (13), choices of the matrix $A^{(1)}$ may give single-valued wave function. However, all of them are not normalizable.

Summarizing, we are led to conclude that the physical spectrum of q_3 for $h = 1/2$ lies on the imaginary axis. The first three levels are quoted in Table I.

It is very instructive to superimpose this result on our earlier calculations, based on a different approach (Bethe ansatz), which results in the analytic expression for the eigenenergy of the three Reggeon system as a function of h and q_3 [10]. Figure 2 shows $\epsilon_3(1/2, q_3)$ along the imaginary axis of q_3. Black dots and crosses mark values of q_3 quantized according to the first condition in (17). It turns out that the candidates which were eliminated by the second condition (crosses) are numerically very close to the poles of the ϵ_3. They are, however, nonphysical since the corresponding wave functions are not single valued.

The intercept of the odderon trajectory is determined by the largest eigenvalue $\epsilon_3(1/2, q_3^O)$. This corresponds

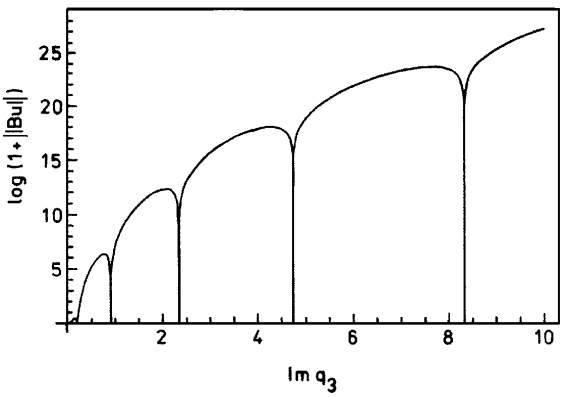

FIG. 1. Quantization of $\hat{q}_3$. Only half of the zeros shown constitutes the physical spectrum.

TABLE I. Quantization of q_3 and corresponding eigenvalues of the holomorphic Hamiltonian.

No.	q_3	ϵ_3
1	$0.20526i$	-0.49434
2	$2.34392i$	-5.16930
3	$8.32635i$	-7.70234

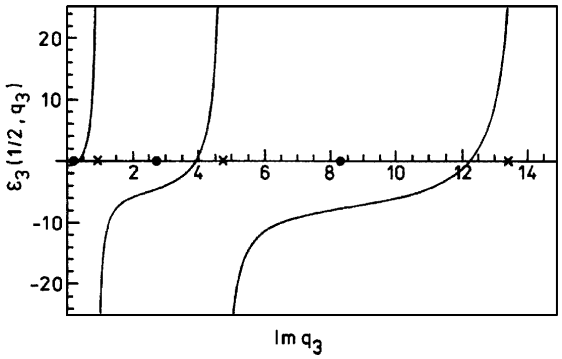

FIG. 2. The holomorphic energy of the three Reggeized gluons for imaginary q_3 (solid lines [10]). Dots and crosses show solutions of the first condition (17). Solutions close to the poles of ϵ_3 are eliminated by the second condition.

to the first nonzero $q_3^O = q_3^{(1)}$ on the imaginary axis, with the numerical value

$$q_3^O = -0.20526i\,, \tag{20}$$

which, together with our solution of the Baxter equation, $\epsilon_3(h, q_3)$ [10], gives, for the energy of the odderon state,

$$\epsilon_3(1/2, q_3^O) = -0.49434\,. \tag{21}$$

This translates for the intercept of the odderon trajectory, c.f. Eq. (1),

$$\alpha_O(0) = 1 - 0.24717\,\frac{\alpha_s N_c}{\pi}\,, \tag{22}$$

which may solve the long-standing phenomenological puzzle of why the odderon trajectory is so hard to observe experimentally. However, any phenomenological consequences of this result should be taken with great caution. The general assumptions behind the derivation of (22) are the same as in the classic work of [1]. In addition, we rely on the assumptions of [10], e.g., analyticity of the solution of the Baxter equation $Q_3(\lambda)$, together with the particular form (2) of [10] for the eigenenergy generalizing known expressions used in the Bethe ansatz approach.

In Table I we quote the first few quantized values of q_3, together with corresponding energies. Indeed the next states have a substantially smaller intercept and, consequently, their contribution to the high energy scattering is negligible.

Our method also provides explicit expressions for the wave functions of the compound states. At the eigenvalues of $\hat{q}_3$ the coefficients of the expansion (12) are given by the common eigenvector corresponding to the zero eigenvalue of $\mathcal{B}_U$ or $\mathcal{B}_L$. Therefore the wave functions are given explicitly, in terms of known bases, and can be used for various applications. For the odderon state we obtain

$$\alpha^O = 0.7096, \qquad \beta^O = -0.6894, \qquad \gamma^O = 0.1457\,. \tag{23}$$

Note that the asymptotic form of the wave function at $z = 1$, implied by the uniqueness condition (13), agrees with that derived by Lipatov from the symmetry considerations [3]. Moreover, it follows from Eqs. (15) and (16) that the same asymptotics hold around other singular points.

A variational estimate of the lower bound for the odderon $\alpha_O > 1 + 0.36\alpha_s N_c/\pi$ was derived in Ref. [4]. Recently this has been challenged by Braun, who gives the bound $\alpha_O(0) > 1 - 0.339\alpha_s N_c/\pi$ [13]. The latter estimate is consistent with our exact result. It would be interesting to repeat their variational calculation with our exact wave function.

Recently Korchemsky studied the dependence of the eigenvalues of $\hat{q}_3$ on h [9]. His results should agree with ours for higher states. In fact, we have found earlier that his WKB formulas reproduce exact results quite well even at low values of q_3.

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Note added.—After we announced these results in hep-th/9802100, Braun, Gauron, and Nicolescu have indeed used our wave function (12), with (13) and (23), in their variational approach (hep-ph/9804432; hep-ph/9809567). They have reproduced numerically our eigenvalue (21), which confirms the results reported here and the method of [10].

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